Arbitrary objects in mathematics and semantics

We propose that an ontology of arbitrary objects, such as the arbitrary prime number, does great service to the philosophy of mathematics and formal linguistics. We offer an alternative foundational set theory for the treatment of arbitrary objects. We claim that this alternative set theory provides a better analysis of the content of mathematical discourse, and opens up a new line of approach to natural language semantics.

Traditionally, natural language semantics and the formalisation of mathematical discourse is carried out in the familiar Zermelo-Fraenkel set theory (**ZF**). Our proposed alternative is Fraenkel-Mostowski set theory (**FM**), developed in the 1930s. FM set theory allows us to give direct semantics to (unbound) variables, variable binding, and substitution. We show how FM techniques help to develop a semantics of arbitrary objects and quantification. We claim this allows for more natural interpretations of the languages used by mathematicians.

We suggest how FM set theory provides a more hospitable foundational universe for semantic treatments of natural language constructions. In particular, it can account for natural language constructions which are difficult to account for using the theory of binding and quantification associated with higher order logic and its traditional set-theoretic model theory in ZF.

Arbitrary objects and mathematics

Mathematical language and reasoning contains abundant apparent reference to arbitrary objects. For example:

- Let x be a positive integer, then it is either odd or even.

We can envisage a naive treatment of such a sentence which interprets x as referring to a special sort of 'arbitrary' object. This would yield a simple account of our knowledge and understanding of general mathematical facts: our knowledge and understanding of general mathematical facts is like our knowledge and understanding of particular facts, except that the particular objects happen to be 'general' objects!

The coherence of such a treatment has been disputed historically by Berkeley and later by Frege. Indeed, Frege's foundational work in mathematical logic was in partly intended as a solution to the problem of general mathematical knowledge that makes no use of arbitrary objects.

Work by Kit Fine in the 1980s has shown that the naive notion of arbitrary objects is formally coherent. However, Fine's treatment of arbitrary objects falls short on two counts. Firstly, there are examples of arbitrary reference that cannot be handled by Fine's account. Secondly, Fine's treatment does not show how arbitrary objects can be explicitly constructed within a set theory that can be used to carry out higher mathematics.

The formal theory of arbitrary objects

By making minor but very specific technical changes to the ZF universe, Fraenkel-Mostowski set theory was originally developed as a means of proving the independence of the axiom of choice from the other axioms of set theory. In fact, a consequence of the structure of the FM universe is the failure of the axiom of choice. This same structure can be interpreted, we claim, as an explicit set theory of arbitrary objects.

The FM universe contains a range of extra elements that can be regarded as the denotations of variables. That is, using FM set theory we can interpret variables directly in the universe (as arbitrary objects), without making any recourse to variable assignments as in the familiar Tarski truth definition.

For example, in ZF set theory an open arithmetic sentence $A(x)$ where only x is free, can receive as its denotation the set S of all numbers such that $v(A) = \top$ where v is any variable assignment. Using Fraenkel-Mostowksi set theory $A(x)$ can receive as its denotation an FM set S_x. A relation Sub definable in FM set theory, modelling substitution, then relates S_x to all the numbers satisfying A with an arbitrary element $[\![x]\!]$ directly interpreting the variable x.

In this way, we obtain a semantics for mathematical discourse where the structure of denotations comes much closer to matching the structure of syntax. This is possible with the FM set theoretic ontology of arbitrary objects interpreting variables.

Mathematical languages. In mathematical languages like first-order logic, arbitrary objects appear as *quantified variables*. Attempts to interpret quantifiers $\forall$ and $\exists$ (as in $\forall x A(x)$ and $\exists x A(x)$) as operations on only a denotation for $A(x)$, are fraught with difficulty. The innovation of Tarski semantics was to circumvent these difficulties by interpreting quantifiers as operations on many denotations for $A(x)$, depending on different variable assignments for x.

Using the model of arbitrary objects given by FM, we can interpret open terms naively as FM sets, and we can interpret the quantifiers naively as operations involving the arbitrary objects within those FM sets. For example, if the denotation of $A(x)$ is S_x then the denotation of $\forall x A(x)$ is the FM set S' where S' is calculated from S_x, using Sub.

Natural languages. Apparent reference to arbitrary objects is not confined to mathematical languages. Natural language expressions apparently refer to arbitrary objects, or at least to entities with no specific reference:

- Suppose *someone* is next door, then *they* are either dead or very quiet.
- If *a farmer* has a donkey then *he* beats it.
- If I have *a pound*, then I'll put *it* in the parking meter.

Our ontology of arbitrary objects goes a long way to providing a face-value semantics for such sentences. We suggest that there are many additional applications of FM to formal semantics, for example we argue it will provide a simple semantics for binding theory in general.

Summary.

Fraenkel-Mostowski set theory provides a more flexible universe for developing formal semantics than that provided by ZF set theory. Because it is sets-based, it is very flexible, just like ZF. It provides a sets universe in which arbitrary objects have direct existence, and so is more flexible than ZF, making it possible to interpret arbitrary objects as they frequently appear both in formal and natural languages.