Consistency of Quine’s New Foundations using nominal techniques

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We build a model in nominal sets for TST+; typed set theory with typical ambiguity. It is known that this is equivalent to the consistency of Quine’s New Foundations.

Permutation methods based on nominal techniques are used to constrain the size of powersets and thus model typical ambiguity.

Additional Key Words and Phrases: Set theory, New Foundations, nominal techniques

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1. INTRODUCTION

Consider the following false reasoning: define \( x = \{ a \mid a \notin a \} \). It is easy to check that \( x \in x \) if and only if \( x \notin x \). This is Russell’s paradox and is one of the central paradoxes of (naive) set theory.

Zermelo-Fraenkel set theory (ZF) avoids paradox by insisting that \( a \) be guarded; we can only form \( \{ a \in y \mid a \notin a \} \) where \( y \) is already known to be a set. The price we pay for this is that we cannot form ‘reasonable’ sets such as the universal set \( \{ a \mid \top \} \) (the set of all sets) or the set of ‘all sets with 2 elements’, and so on. In ZF, these are proper classes (a nice historical account of Russell’s paradox is in [Gri04]; for ZF see e.g. [Jec06]).

New Foundations (NF) avoids paradox by insisting on a stratifiable language [Qui37]. Every variable and term can be assigned a level, such that we only form \( t \in s \) if \( \text{level}(s) = \text{level}(t)+1 \). So \( a \in a \) and \( a \notin a \) are outlawed because no matter what level \( i \) we assign to \( a \), we cannot make \( i \) equal to \( i+1 \). We can stratify \( \top \) so we can still form the universal set in NF, and ‘has 2 elements’ is also stratifiable.

Excellent discussions are in [For95] and [Hol98], and a clear summary with a brief but well-chosen bibliography is in [For97].

However, at the time of writing we know of no published proof of consistency for NF (relative e.g. to ZF). This has been the situation since NF was introduced in 1937 in [Qui37].

This paper presents what the author believes to be a full proof of the consistency of NF.

Note that this is a paper about NF; it is not a paper in NF. Familiarity with NF, TST+, or TZT+ as reasoning systems and foundations of mathematics, with all their unique and special features, is not relevant to understanding this paper.

1.1. How this paper works

What follows is intended to give some overall feeling for how the proofs fit together and is not intended as an exhaustive description of the technical detail.

First, NF is equiconsistent with TST+ [Spe62]. The proof in this paper is for consistency of TST+; consistency of NF is a corollary. The impatient reader can now jump straight to Definition 4.3, which presents the restricted powerset model on which this proof is based, then consult Figure 5 and Theorem 6.35 which present the relevant bijection, and finally look at Proposition 7.5 which is where the bijection is applied.

The syntax of TST+ is the language of first-order logic extended with base predicates for sets equality \( s=t \) and sets epsilon \( t \in s \). Also, terms have sets comprehension \( \{ a \mid \phi \} \).

See Figure 1 for the full syntax, and axioms are listed in Figures 2 and 3. There is no need to list them here because they are mostly as one might expect of a set theory. There are two non-obvious features:

— Variable symbols \( a \) are assigned levels, as mentioned above. Levels are natural numbers; levels extend to terms and each term in TST+ syntax has a fixed level (in NF, levels may vary). The reader can think of levels as types.
\( \phi \) is subject to a \textit{stratification} typing condition that we may only form \( t \in s \) in \( \phi \) if \( \text{level}(s) = \text{level}(t) + 1 \). Stratification is an easy-to-express and decidable syntactic condition which cuts down on the well-formed terms and predicates.

— TST+ is an extension of TST by typical ambiguity.

The obvious way to proceed is to choose a set \( U \) to denote level 0, and to denote level \( i + 1 \) with the powerset of the denotation of level \( i \). Sets are denoted by sets, and predicates by truth-values.

Call this the \textbf{sets and powersets} semantics of TST (see Remark 2.9). The problem with the sets and powersets semantics is that it is unsound for typical ambiguity.

A minimum change to the sets and powersets semantics so that it might become sound is as follows:

1. We could try to restrict powersets to keep the same cardinality as we ascend levels—while still preserving enough elements to interpret the language of TST+.
2. We could then biject powersets of different levels.

It may seem that step 2 above follows immediately from step 1: if the sets have the same cardinality then of course they can be bijected (see Theorem 5.1). This is true but a simple pointwise bijection fails due to diagonalisation arguments; if we are to biject levels, compatibly with sets extensionality and comprehension, then we need something more sophisticated. So these two steps above really are distinct, and this will be reflected in the maths that will follow.

We will use two similar but distinct pieces of technology, which we can write as \( E \$ x \) (which we use to restrict cardinalities of powersets) and \( S \$ x \) (which we use to biject different levels), and they interact in interesting ways.

1. We control the size of powersets using a notion of support written \( E \$ x \), where \( E \) is a small equivalence relation.\(^1\)
   
   The main definitions are Definitions 3.20 and 4.3; the main result is Theorem 5.1.
2. We move between levels using another notion of support written \( T_k \$ x \) and \( S_k \$ x \). Here \( T_k \) and \( S_k \) are large sets of elements.\(^2\)
   
   The main definitions are Definitions 3.24, 6.4, and 6.12. The interesting interaction between our two notions of support happens in Theorem 5.20, which is a key technical result and enables a technical development culminating with Theorem 6.35. A key technical device is \( \rho \) from Definition 3.29, the background to which is discussed in Remark 3.34. See also the discussion in Remark 6.1.

The concluding argument is Proposition 7.5 and Corollary 7.8. Theorem 8.1 then quickly follows.

### 1.2. Further comments

**Remark 1.1.** As discussed, this paper proves consistency of TST+ (typed set theory with typical ambiguity) relative to ZF sets (Zermelo-Fraenkel set theory); consistency of NF follows since NF is known to be consistent relative to TST+ [Spe62].

**Remark 1.2.** This paper proves consistency of NF by by using—and by rather considerably developing on—\textit{nominal techniques}, which were originally based on Fraenkel-Mostowski set theory (\textit{FM}), itself based on Zermelo-Fraenkel set theory with atoms (\textit{ZFA}).

Familiarity with these set theories is not necessary to understand the body of the paper. Accounts of them tailored specifically to nominal techniques and mostly compatible with the notations and conventions of this paper, appear in [Gab01; Gab11; DG12]. A linkage of some ‘nominal’ ideas to some corresponding ‘Fraenkel-Mostowski sets’ ideas is given in [Gab11, Remark 2.22].

In any case: this paper is self-contained.

---

\(^1\)Experts in nominal techniques note: \( E \) is not a small set of atoms; it is an equivalence relation with a small number of equivalence classes.

\(^2\)‘Small’ and ‘large’ have technical meanings. See Notation 3.12.
Remark 1.3. FM set theory was originally developed to prove the independence of the Axiom of Choice from the other axioms of set theory, by building a universe in which a ‘hereditary support’ axiom holds that contradicts Choice.

The ideas of this paper build on nominal techniques, which build on FM set theory. And indeed, this paper is positively brimming with nominal-style support properties (Subsection 3.4 would be a good place to start reading). They exist in considerable variety and are used everywhere.

Yet: this paper uses also Choice (for just one example, see the proof of Proposition 5.24). How can this be?

The point is, that in this paper (and in most of this author’s other publications) we use nominal-style support conditions not as axioms, but as well-behavedness conditions. Some sets in this paper are (say) supported by $T_k$ for some $k < \omega$, or supported by $E$ for some $E \in EQ_i$—and some are not. Choice functions generally are not well-supported, but we will not expect them to be.

Remark 1.4. The need for two distinct notions of support seems to have to do with avoiding diagonalisation arguments. The following discussion is highly informal, but may be helpful:

Think of diagonalisation as measuring a form of entropy; so that a diagonalisation arguments proves that as we go up a level in a powerset hierarchy, entropy increases. This entropy usually takes the form of cardinality (i.e. taking a powerset increases cardinality) but if our powersets are hereditarily supported in some way, in nominal style, then entropy may instead take the form of an asymmetry (i.e. taking a powerset causes support conditions to fail).

In any case: as we ascend the powerset hierarchy, whatever measure of entropy we use is guaranteed to increase, and whatever restrictions on symmetry or cardinality that we try to impose, will be eroded. However, if we have two notions of support, each one sensitive to one kind of entropy but not the other, then we can control cardinality using one notion, and control asymmetry using the other. By this view, the ‘squashing’ result Theorem 5.20 is a mechanism for discharging entropy that may have accumulated moving from level 1 to level 2.

More on this in Remark 6.1, by which time we will have built the machinery to make this discussion more formal.

Remark 1.5. Readers coming to this paper from a theoretical computer science background might like to think of the paper as follows: TST+ is a simply-typed $\lambda$-calculus where $\{a | \phi\}$ corresponds to $\lambda a.\phi$ and $t \in s$ corresponds to $s t$ ($s$ applied to $t$). There are some natural axioms for first-order logic and axioms corresponding to $\beta$- and $\eta$-conversion, given in Figure 2; and one unusual axiom corresponding to typical ambiguity, given in Figure 3. It is very simple to specify, if difficult to prove consistent.

2. TST AND TST+

In this Section we set up TST and TST+. We do not yet have much machinery to prove anything about them, aside from to observe why the sets and powersets model is insufficient (Remark 2.9).

2.1. Basic syntax

We define the syntax of typed set theory (TST):

Definition 2.1. (1) For every $i \geq 1$ choose a disjoint countably infinite set of variable symbols $\text{Var}_i$.

If $a \in \text{Var}_i$, then define $\text{level}(a) = i$.

(2) For every $i \geq 1$ assume we are given a disjoint set of constants $V_i$. The notation clash with $V_i$ from Definition 4.3 is deliberate – but for now $V_i$ is just a set of constant symbols.

3It might be worth making an analogy: From the point of view of group theory, invertibility is an axiom. From the point of view of semigroup theory, invertibility is a well-behavedness condition that some elements have and others do not. The potential for confusion arises only if somebody reads a paper on semigroups under a background assumption that it is a paper on groups.
\[ s, t \in \text{Term}_i ::= a \in \text{Var}_i \mid c \in V_i \]

\[ \phi, \psi \in \text{Pred} ::= \bot \mid \phi \land \psi \mid \neg \phi \mid \exists a. \phi \mid s = t \mid t \in s \]

**Fig. 1:** The syntax of typed set theory (Definition 2.1)

If \( c \in V_i \) then define \( \text{level}(c) = i \).

1. Define \( (\text{level } i) \) terms \( s, t \in \text{Term}_i \) and predicates \( \phi, \psi \in \text{Pred} \) inductively as in Figure 1. The predicate \( s \in t \) is subject to a side-condition that \( s \) and \( t \) should have the same level; the predicate \( t \in s \) is subject to a side-condition that the level of \( s \) should be the level of \( t \) plus 1.

2. The variable symbol \( a \) is bound in \( \exists a. \phi \), as usual. We identify terms up to \( \alpha \)-equivalence as usual.

3. We write \( \text{fv}(\phi) \) and \( \text{fv}(s) \) for the free (unbound) variable symbols in \( \phi \) and \( s \) respectively, and \( \text{consts}(\phi) \) and \( \text{consts}(s) \) for the constants that appear in \( \phi \) and \( s \) respectively.

To \( \phi = \emptyset \) then we call \( \phi \) closed, and similarly for \( s \). Write

\[ \text{CPred} = \{ \phi \in \text{Pred} \mid \text{fv}(\phi) = \emptyset \}. \]

So \( \text{CPred} \) is the set of closed predicates. Note that closed predicates may still mention constants.

4. If \( a \in \text{Var}_i \) and \( s \in \text{Term}_i \) then we write \( [a := s] \) for capture-avoiding substitution over predicates and terms as usual, as in \( \phi[a := s] \).

**Remark 2.2.** We intend that:

— The intuition of \( t \in s \) is ‘the value denoted by \( t \) is an element of the value denoted by \( s \).’

— The intuition of \( s = t \) is ‘\( s \) denotes the same value as \( t \).’

— The intuition of \( \exists a. \phi \) is ‘\( \phi \) holds for some possible value for \( a \).’

These intuitions will be made precise in Figure 4.

**Notation 2.3.** Suppose \( \phi, \psi \in \text{Pred} \). We define syntactic sugar:

\[
\begin{align*}
\top & = \neg \bot \\
\phi \lor \psi & = \neg (\neg \phi \land \neg \psi) \\
\forall a. \phi & = \neg \exists a. \neg \phi \\
\phi \Rightarrow \psi & = (\neg \phi) \lor \psi \\
\phi \Leftrightarrow \psi & = (\phi \Rightarrow \psi) \land (\psi \Rightarrow \phi)
\end{align*}
\]

We may simply abuse notation ignore that these are not primitive syntax.

Definition 2.4 will be helpful later, starting from Subsection 7.1:

**Definition 2.4.** Define \( \text{minlev}(\phi) \) the **minimum level** of \( \phi \) inductively by:

\[
\begin{align*}
\text{minlev}(a) & = \text{level}(a) \\
\text{minlev}(\phi \land \psi) & = \min(\text{minlev}(\phi), \text{minlev}(\psi)) \\
\text{minlev}(\exists a. \phi) & = \min(\{\text{level}(a), \text{minlev}(\phi)\}) \\
\text{minlev}(c) & = \text{level}(c) \\
\text{minlev}(\neg \phi) & = \text{minlev}(\phi) \\
\text{minlev}(t \in s) & = \min(\text{level}(t), \text{level}(s))
\end{align*}
\]

**2.2. Axioms of TST and TST+**

**Definition 2.5.** Fix a \( \vartheta \) \( \downarrow \) map

\[ \vartheta : \bigcup_{i \geq 1} \text{Var}_i \cong \bigcup_{i \geq 1} \text{Var}_i \]

which bijects \( \text{Var}_i \) with \( \text{Var}_{i+1} \) for each \( i \geq 1 \).

We can do this because we assumed in Definition 2.1 that variable symbols are countably infinite at every level.

**Definition 2.6.** Extend \( \vartheta \) to an action on \( \phi \in \text{Pred} \) such that \( \text{consts}(\phi) = \emptyset \) (so \( \phi \) is a predicate that may have free variables but must mention no constants), and on \( s \in \text{Term}_i \) such that \( \text{consts}(s) = \emptyset \) (so \( s \) is a term that may have free variables but must mention no constants), in the natural way:

\[
\begin{align*}
\vartheta(\bot) & = \bot \\
\vartheta(\exists a. \phi) & = \exists \vartheta(a), \vartheta(\phi) \\
\vartheta(s \in t) & = (\vartheta.s) \in (\vartheta.t) \\
\vartheta(\neg \phi) & = \neg \vartheta(\phi) \\
\vartheta(\text{level}(t)) & = \text{level}(\vartheta(t)) \\
\vartheta(s \in t) & = (\vartheta.s) \in (\vartheta.t)
\end{align*}
\]
(false) \[ \vdash \bot \Rightarrow \phi \]
(modus ponens) \[ \text{If } \vdash \phi \text{ and } \vdash \phi \Rightarrow \psi \text{ then } \vdash \psi \]
(generalisation) \[ \text{If } \vdash \phi \text{ then } \vdash \forall a. \phi \]
(K) \[ \vdash \phi \Rightarrow (\psi \Rightarrow \phi) \]
(S) \[ \vdash ((\phi \Rightarrow \psi) \Rightarrow \xi) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \xi)) \]
(contrapositive) \[ \vdash (\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi) \]
(instantiation) \[ \vdash (\forall a. \phi) \Rightarrow ((\phi[a := s]) \Rightarrow \psi) \]
(allR) \[ \vdash (\forall a. (\phi \Rightarrow \psi)) \Rightarrow ((\psi \Rightarrow \phi \Rightarrow \forall a. \psi)) \quad a \text{ not free in } \phi \]
(identity) \[ \vdash s = s \Rightarrow (\phi[a := s] \Leftrightarrow \phi[a := t]) \]
(Leibniz) \[ \vdash s = t \Leftrightarrow \forall c. (c \in s \Leftrightarrow c \in t) \quad c \text{ not free in } s, t \]
(comprehension) \[ \vdash \exists c. (c \in s \Leftrightarrow \phi[a := s]) \]

**Fig. 2:** Axioms of TST

\[ \vdash \phi \Leftrightarrow \theta \cdot \phi \quad \text{consts}(\phi) =fv(\phi) = \emptyset \]

**Fig. 3:** Additional *typical ambiguity* axiom of TST+

**Definition 2.7**(1) Define a derivability relation \( \vdash \phi \) for *typed set theory* (**TST**) by the rules in Figure 2.

(2) Define a derivability relation, also written \( \vdash \phi \), for *typed set theory with typical ambiguity* (**TST+**) by the rules in Figure 2 along with the additional axiom-scheme in Figure 3.

**Remark 2.8.** \( \phi \) is assumed to be closed, and we identify syntax up to \( \alpha \)-equivalence, so the effect of \( \theta \cdot \phi \) is merely to shift the levels of the variable symbol in \( \phi \)—which are all bound—up by one.

**Remark 2.9.** It is easy to prove TST consistent: it suffices to choose a set \( U \) to denote level 0, and to denote level \( i+1 \) with the powerset of the denotation of level \( i \).

Given a valuation for the variable symbols, sets are denoted by sets and predicates by truth-values: it is routine to check that all the axioms in Figure 2 are valid.

Call this the *sets and powersets* semantics of TST.

From the point of view of the sets and powersets semantics of TST, the difficulty is typical ambiguity in Figure 3. The rest of this paper is devoted to constructing a sets and powersets semantics of TST+.

### 3. SOME BASIC CONSTRUCTIONS

#### 3.1. Ordinals, powersets, and cardinalities

We set up some standard background theory:

**Definition 3.1.** Write \( \text{ON} \) for the class of ordinals. This can be taken to be the least collection of sets such that:

(1) \( \text{ON} \) is **transitive**, meaning that if \( \alpha \in \text{ON} \) and \( \alpha' \in \alpha \) then \( \alpha' \in \text{ON} \).
(2) If \( U \) is a transitive subset of \( \text{ON} \), then \( U \in \text{ON} \).

**Definition 3.2.** Write \( \mathbb{N} \subseteq \text{ON} \) for the set of natural numbers \( \{0, 1, 2, \ldots \} \). Note that \( \mathbb{N} \in \text{ON} \).

When viewed as an ordinal \( \mathbb{N} \) is usually written \( \omega \); we may use \( \mathbb{N} \) and \( \omega \) synonymously.

**Definition 3.3.** Write \( \text{pow}(X) \) for the powerset of \( X \), that is:

\[ \text{pow}(X) = \{ X' \mid X' \subseteq X \} \].
Extend $\text{pow}(X)$ to an iterated operation $\text{pow}^\alpha(X)$ for $\alpha \in \text{ON}$ by ordinal induction:

\[
\begin{align*}
\text{pow}^0(X) &= X \\
\text{pow}^{\alpha+1}(X) &= \text{pow}(\text{pow}^\alpha(X)) \quad \alpha \in \text{ON} \\
\text{pow}^\lambda(X) &= \bigcup_{\alpha < \lambda} \text{pow}^\alpha(X) \quad \lambda \in \text{ON} \text{ a limit ordinal}
\end{align*}
\]

**Definition 3.4**

1. We write $\#X \in \text{ON}$ for the cardinality of $X$, which we take to be the least ordinal that bijects with $X$.
2. If $C \in \text{ON}$ is a cardinality then we write $2^C$ for the cardinality of $\text{pow}(C)$.
3. If $\alpha \in \text{ON}$ define $\beth(\alpha)$ by
   \[ \beth(\alpha) = \#(\text{pow}^\alpha(\mathbb{N})). \]

**Remark 3.5.** So intuitively:

1. $\beth(0)$ is the cardinality of $\mathbb{N}$.
2. $\beth(1)$ is the cardinality of $\text{pow}(\mathbb{N})$.
3. $\beth(\alpha)$ is the cardinality of the $\alpha$-th powerset of $\mathbb{N}$.

**Notation 3.6.** If $\alpha \in \text{ON}$ then write $\mathcal{N}(\alpha)$ for the $\alpha$-th cardinality above that of $\mathbb{N}$. So $\mathcal{N}(0) = \beth(0)$ and $\mathcal{N}(\alpha) \leq \beth(\alpha)$ for every $\alpha \in \text{ON}$.

Define the **generalised continuum hypothesis** (GCH) to be the assertion that there are no ‘extra’ cardinalities in-between the cardinalities of powersets:

\[ (\text{GCH}) \quad \forall \alpha \in \text{ON}. (\mathcal{N}(\alpha) = \beth(\alpha)). \]

**Lemma 3.7.** Given the GCH, there are $\omega$ many cardinalities below $\beth(\omega)$.

**Proof.** Immediate from GCH. \qed

**Definition 3.8.** We assume GCH henceforth (more on this in Remark 8.2).

### 3.2. (Small) equivalence relations

This is the first slightly nontrivial subsection: we introduce the notions of small and large set, and of a small equivalence relation.

**Definition 3.9.** Suppose $X$. Define the sets of relations $\text{Rel}(X)$ and equivalence relations $\text{Equiv}(X)$ on $X$ as standard by:

\[ \begin{align*}
\text{Rel}(X) &= \text{pow}(X \times X). \\
X &\in \text{Equiv}(X) \text{ when } X \in \text{Rel}(X) \text{ and } X \text{ is reflexive, transitive, and symmetric.}
\end{align*} \]

**Definition 3.10.** Suppose $X$ is a set. Equivalence relations $E, E' \in \text{Equiv}(X)$ are partially ordered $E \leq E'$ in a natural way as follows:

\[ E \leq E' \quad \text{when} \quad \forall x, x' \in X. ((x, x') \in E \Rightarrow (x, x') \in E'). \]

(So $E \leq E'$ when $E$ is at least as fine as $E'$.)

**Definition 3.11.** Suppose $X$ is a set and $E, E' \in \text{Equiv}(X)$. Write $E \wedge E'$ for the greatest equivalence relation contained in both $E$ and $E'$. More concretely:

\[ E \wedge E' = \{(x, x') \mid (x, x') \in E \wedge (x, x') \in E'\}. \]

We will use Notation 3.12 repeatedly in this paper; we mention it now:

**Notation 3.12.** Suppose $X$ is a set.

\[ \begin{align*}
&\text{If } \#X < \beth(\omega) \text{ then call } X \text{ small.} \\
&\text{If } \#X = \beth(\omega+1) \text{ then call } X \text{ large.}
\end{align*} \]

An immediate abuse of Notation 3.12 will be helpful:
Notation 3.13. Suppose $X$ is a set and $E \in \text{Equiv}(X)$ is an equivalence relation on it.

(1) If $e \subseteq X$ then write $e \in E$ when $e = \{x' \mid (x', x) \in E\}$ for some $x \in X$. In words, $e$ is an equivalence class in $E$.

(2) Write $\#E$ for

$$\#E = \# \{ e \subseteq X \mid e \in E \}.$$  

In words, $\#E$ is the cardinality of the set of equivalence classes of $E$.

Definition 3.14. Suppose $X$ is a set and $E \in \text{Equiv}(X)$ is an equivalence relation on it. Call $E$ small when

$$\#E < \mathfrak{U}(\omega).$$

Lemma 3.15. Suppose $X$ is a set and $E, E' \in \text{Equiv}(X)$. Then:

(1) $E \land E' \leq E$

(2) $\#(E \land E') \leq \#E \times \#E'$ (where $\times$ here denotes cardinal multiplication).

(3) As a corollary, if $E$ and $E'$ are small then so is $E \land E'$.

Proof. By routine calculations. □

Remark 3.16. So for the case of equivalence relations only:

— $e \in E$ means “$e$ is an equivalence class in $E$” (and not “$e$ is literally an element of $E$ as a set”).

— $\#E$ means “the number of equivalence classes in $E$” (and not “the number of pairs $(x, x')$ in $E$ as a graph”).

— A small equivalence relation is one with a small number of equivalence classes.

Using this notation we can rephrase Definition 3.10 in the following quite nice manner:

$$E \leq E' \quad \text{when} \quad \forall e \in E, \exists e' \in E'. e \subseteq e'.$$

3.3. Symmetry groups and permutations

Nominal techniques are based on sets with a permutation action. We introduce the theory needed to talk about this, including in particular a notion of support $E \$ x$ (Definition 3.20). Though this Subsection is not technically difficult, it is novel within the nominal techniques literature because the notion of support it is based on uses $E$ an equivalence relation on atoms rather than a small set of atoms.

Definition 3.17. Suppose $X$ and $A$ are sets.

(1) The symmetry group or set of permutations $\text{Perm}(A)$ is the set of bijections $\pi$ on $A$.

(2) If $\pi \in \text{Perm}(A)$ then define nontriv($\pi$) by

$$\text{nontriv}(\pi) = \{ a \in A \mid \pi(a) \neq a \}.$$

(3) A permutation action on $X$ over $A$ is a group action of the symmetry group $\text{Perm}(A)$ on $X$; that is, a function $\cdot$ from $\text{Perm}(A) \times X$ to $X$ such that $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ and $\text{id} \cdot x = x$.

Remark 3.18 (Technical note). In nominal techniques, permutations $\pi$ are typically (though not universally) subject to a size restriction that $\pi$ should permute only a small number of elements (where ‘small’ has a precise meaning, typically, though again not universally, linked to a cardinality restriction such as ‘is finite’).

There is no smallness restriction in Definition 3.17, and furthermore this will be important. We will need permutations that can shift nearly all of $A$ at once, for instance: in the case of $X \in V_2$ of the proof of Theorem 5.1, we need $\pi \in \text{stab}(E_0)$ to be possibly large for the counting arguments to work; and in Lemma 5.18 we need be able to biject $V_2 \setminus S_{k+1}$ with $V_2 \setminus S_k$. 

Definition 3.19. Suppose $X$ and $A$ are sets and · is a permutation action on $X$ over $A$. Suppose $i > 0$. Then

— $i$-fold powersets $X \in \text{pow}^i(X)$,
— relations on them $E \in \text{Rel}(\text{pow}^i(X))$, and
— equivalence relations on them $E \in \text{Equiv}(\text{pow}^i(X))$

inherit the permutation action pointwise in a standard way:

$$\pi \cdot X = \{\pi \cdot x \mid x \in X\}$$

$$\pi \cdot E = \{(\pi \cdot X, \pi \cdot X') \mid (X, X') \in E\}$$

(We should check that if $E$ is an equivalence relation then so is $\pi \cdot E$, but this is very easy.)

We use $E \$ x$ in Definition 3.20 from Section 4 onwards, most notably in Definition 4.3:

Definition 3.20. Suppose $X$ is a set with a permutation action over a set $A$. Suppose $E \in \text{Equiv}(A)$ is an equivalence relation on $A$.

(1) Define $\text{stab}(E)$ by

$$\text{stab}(E) = \{\pi \in \text{Perm}(A) \mid \forall a \in A. (a, \pi(a)) \in E\}.$$ 

We may call a permutation $\pi \in \text{stab}(E)$ a $E$-permutation, and we may say that $\pi$ is an $E$-permutation when it respects the equivalence classes in $E$, by only permuting elements within each equivalence class.

(2) Define $E \$ x$, and write $E$ supports $x \in X$, as follows:

$$E \$ x \quad \text{when} \quad \forall \pi \in \text{stab}(E). \pi \cdot x = x.$$ 

(3) Call $x \in X$ simple (over $A$) when it has a small (Definition 3.14) supporting $E \in \text{Equiv}(A)$.

Lemmas 3.21 and 3.22 are useful sanity checks:

Lemma 3.21. Suppose $X$ is a set with a permutation action over a set $A$ and suppose $E, E' \in \text{Equiv}(A)$ and $x \in X$ and $E \leq E'$ (Definition 3.10; so $E$ is a finer relation than $E'$). Then

$$E' \$ x \implies E \$ x.$$ 

Proof. By routine calculations from Definitions 3.20 and 3.10.

Lemma 3.22. Suppose $X$ is a set with a permutation action over a set $A$ and suppose $E, E' \in \text{Equiv}(A)$ and $x \in X$. Then

$$E \$ x \implies (E \land E') \$ x.$$ 

Proof. We combine Lemmas 3.15(1) and 3.21.

Definition 3.23 (Natural and pointwise actions). Every set $X$ has a natural permutation action over itself, given by

$$\pi \cdot x = \pi(x).$$ 

It follows using Definition 3.19 that for every $i$ and $i'$,

$\text{pow}^i(X)$ and $\text{Rel}(\text{pow}^i(X))$ and $\text{Equiv}(\text{pow}^i(X))$

have natural pointwise permutation actions over $\text{pow}^{i'}(X)$:

— If $i' < i$ then we just note that $\text{pow}^i(X) = \text{pow}^{i-i'}(\text{pow}^{i'}(X))$ and we use the natural permutation action acting pointwise.
— If $i' = i$ then we use the natural permutation action directly.
— If $i' > i$ then we use the trivial action

$$\pi \cdot X = X \quad \text{and} \quad \pi \cdot E = E.$$
3.4. Notions of support

We have from Definition 3.20 a notion of support $E\$x$, denoting that the equivalence relation $E$ supports the element $x$. We need another notion of support (which is closer to the idea of support familiar from [GP01]) written $S\$x$ and defined in Definition 3.24. We use $S\$x$ from Subsection 5.2 onwards, notably in the ‘squashing properties’ of Subsection 5.3.

**Definition 3.24.** Suppose we have the following data:

— $X$ is a set with a permutation action over a set $A$.
— $S \subseteq A$ is any set.

Then we define notation as follows:

1. Define $\text{fix}(S)$ by
   $$\text{fix}(S) = \{ \pi \in \text{Perm} \mid \forall a \in S. \pi(a) = a \}.$$  

2. Suppose $x \in X$. Define $S\$x$, and say that $S$ **supports** $x$, by
   $$S\$x \quad \text{when} \quad \forall \pi \in \text{Perm}(A). (\pi \in \text{fix}(S) \Rightarrow \pi \cdot x = x).$$

   Note that if $X = A$ and $\#(A \setminus S) > 1$ then $S\$x$ if and only if $x \in S$.

3. Suppose $E \in \text{Equiv}(X)$ is an equivalence relation on $X$. Define $S\$E$ by
   $$S\$E \quad \text{when} \quad \forall \pi \in \text{Perm}(A). \forall x, x' \in X. ((x, x') \in E \Rightarrow (\pi \cdot x, \pi \cdot x') \in E).$$

   So $S\$E$ when $S$ supports $E$ in the natural sense derived from part 2 of this definition.

4. Suppose $\pi \in \text{Perm}(X)$ is a permutation on $X$ (not on $A$). Define $S\$\pi$ by
   $$S\$\pi \quad \text{when} \quad \forall x \in \text{nontriv}(\pi). S\$x.$$  

   In the case that $X = A$ and the permutation action is the natural one that $\pi \cdot x = \pi(x)$, then $S\$\pi$ precisely when $\text{nontriv}(\pi) \subseteq S$.

**Lemma 3.25.** Suppose $X$ is a set with a permutation action over a set $A$ and suppose $S \subseteq A$ is any set and $x \in X$ and $\pi \in \text{Perm}(A)$. Then

$$S\$x \quad \text{if and only if} \quad \pi \cdot S\$\pi \cdot x,$$

where $\pi \cdot S = \{ \pi \cdot x \mid x \in S \}$ is the pointwise action from Definition 3.19.

**Proof.** By routine calculations on Definitions 3.24 and 3.19. \hfill \Box

**Lemma 3.26.** Suppose $X$ is a set with a permutation action over a set $A$. Suppose $S, S' \subseteq A$ and $x \in X$. Then if $S\$x$ and $S'\$x$ then $S \cap S'\$x$.

**Proof.** By routine calculations; a clear and concise presentation of the details is in [Gab11, Theorem 2.21] (stated for finite permutations, but this makes no difference); a longer and more general proof is in [Gab07, Lemma 37]. \hfill \Box

**Lemma 3.27.** Continuing the notation of Definition 3.24, suppose $x \in X$ and $X \subseteq X$ and $\pi \in \text{Perm}(A)$ is a permutation.

1. If $S\$x$ and $\pi \in \text{fix}(S)$ then $\pi \cdot x = x$.
2. If $S\$x$ and $\pi \in \text{fix}(S)$ then $x \in X$ if and only if $\pi \cdot x \in X$.

**Proof.** (1) This just restates Definition 3.24(2).

(2) From part 1 of this result, noting of the pointwise action (Definition 3.19) that $x \in X$ if and only if $\pi \cdot x \in \pi \cdot X$. \hfill \Box
Lemma 3.28. Suppose $X$ is a set with the natural permutation action over itself (Definition 3.23), and $E \in \text{Equiv}(X)$ and $e \in E$ is an equivalence class in $E$ and $S \subseteq X$. Then
\[ S \cup e = X \implies S \not\in E. \]

Proof. By routine calculations from the definitions.

3.5. Nominal restriction

If $X \subseteq X$ is not supported by some $A$ then it is possible to in a certain sense make it be supported by $A$, by completing $X$ under the orbit of permutations that fix elements in $A$. This is Definition 3.29; we write it $\rho A.X$.

Definition 3.29. Suppose $X$ is a set with a permutation action over a set $A$. Suppose $A \subseteq A$ and $X \subseteq X$. Define $\rho A.X \subseteq X$ by
\[ \rho A.X = \{ \pi \cdot x \mid x \in X, \pi \in \text{fix}(A) \}. \]

Lemma 3.30. Suppose $X$ is a set with a permutation action over a set $A$. Suppose $A \subseteq A$ and $X \subseteq X$ and $\tau \in \text{Perm}(A)$. Then:

1. $\tau \cdot (\rho A.X) = (\rho \cdot A) \cdot \tau \cdot X$.
   (Here $\tau \cdot A = \{ \tau(a) \mid a \in A \}$ is the pointwise action.)
2. As a corollary, if $\text{nontriv}(\tau) \cap A = \emptyset$ then $\tau \cdot (\rho A.X) = \rho A.X$.

Proof. By routine calculations on groups, using Definition 3.29.

Lemma 3.31. Suppose $X$ is a set with a permutation action over a set $A$. Suppose $A \subseteq A$ and $X \subseteq X$ and $x \in X$ and $A \not\subseteq x$. Then
\[ x \in \rho A.X \iff x \in X. \]

Proof. Suppose $x \in \rho A.X$, meaning using Definition 3.29 that $\pi^{-1} \cdot x \in X$ for some $\pi \in \text{fix}(A)$. By Lemma 3.27(1) (since $A \not\subseteq x$) $\pi^{-1} \cdot x = x$. Thus $x \in X$ as required.

The reverse implication is identical.

The following technical lemma will be useful:

Lemma 3.33. Suppose $X$ is a set with a permutation action over a set $A$. Suppose $A, B, B', C \subseteq A$ and $X \subseteq X$ are such that:

- $A \subseteq B, B' \subseteq C$ (we do not require $B \subseteq B'$).
- $\#(B \setminus A) = \#(B' \setminus A) = \#(C \setminus B) = \#(C \setminus B') = \#(A \setminus A)$.
- For every $\pi \in \text{Perm}(A)$, if $\text{nontriv}(\pi) \subseteq C \setminus A$ then $\pi \cdot X = X$.

Then
\[ \rho B.X = \rho B'.X. \]

Proof. By concrete calculations, noting that because of our cardinality assumptions there exists a permutation $\pi$ such that $\text{nontriv}(\pi) \subseteq C \setminus A$ (so that $\pi \cdot X = X$), and $\pi \cdot B = B'$. 

\[ Q.E.D. \]
Remark 3.34. $\rho A.X$ is an interesting construction so we take a moment to place it in the context of some of the nominal literature. The nominal atoms-abstraction $[a].x$ which first motivated nominal techniques in [GP01] can be viewed as a special case of $\rho$ by setting (in the notation of [GP01]) $[a].x = \rho(supp(x)\{a\}).\{(a, x)\}$. The construction $\rho A.X$ powers the detailed study of the fine structure of nominal sets in [Gab09]. More abstract studies of $\rho$ and related sets operation are in [GC11, Definition 4.1] or [Gab11, Definition 9.34]; by this view, Theorem 6.35 is a descendent of Theorem 9.42 from [Gab11].

The $\rho$ in this paper elaborates on previous work in some respects: we are restricting over multiple sets of permutable elements (the $V_i$), and are doing so in a permissive nominal context (see Remark 5.7).

4. THE HIERARCHY OF VERY SIMPLE SETS AND EQUIVALENCE RELATIONS

4.1. The hierarchy

We are now ready to set up the hierarchy of powersets within which we will ultimately find a model of TST+.

4.1.1. The definition

Definition 4.1. Fix a set of atoms $A$ with $\#A = \beth(\omega)$ (Definition 3.4).

Remark 4.2. In Definition 4.1 it would suffice to take $A = \text{pow}^\omega(N)$, but we shall never use the internal structure of an atom.

Note that the atoms of Definition 4.1 are not connected to the urelemente in NFU (NF with urelemente) [Hol98].

Definition 4.3. Define hierarchies $V_i$ and $\text{EQ}_i \subseteq \text{Rel}(V_i)$ of very simple sets and (equivalence) relations inductively using the natural permutation action from Definition 3.23 as follows:

1. $V_0 = A$.
2. $E \in \text{EQ}_0$ when:
   - (a) $E \in \text{Equiv}(A)$ (so $E$ is an equivalence relation on atoms).
   - (b) $\#E < \beth(\omega)$.
3. $X \in V_{i+1}$ when:
   - (a) $X \subseteq V_i$.
   - (b) For every $0 \leq i' < i$ there exists $E_{i'} \in \text{EQ}_{i'}$ such that $E_{i'} \subseteq X$.
4. $E \in \text{EQ}_{i+1}$ when:
   - (a) $E \in \text{Equiv}(V_{i+1})$.
   - (b) $\#E < \beth(\omega)$.
   - (c) For every $0 \leq i' \leq i$ there exists $E_{i'} \in \text{EQ}_{i'}$ such that $E_{i'} \subseteq E$.

Remark 4.4. In condition 3b of Definition 4.3 we insist on $i' < i$ only because the case $i' = i$ is guaranteed: if $i' = i$ then $X \subseteq V_i$ is supported by $\{X, V_i \setminus X\}$. The strict inequality in 3b saves us writing a few trivial cases later on.

If $i = 0$ then this condition is trivially satisfied, so $X \in V_1$ is just a set of atoms.

Notation 4.5. Recall the syntax of predicates and terms from Definition 2.1. Write $\text{CPred}$ for the syntax where the set of level $i$ constants is taken to be $V_i$ for each $i \geq 1$. We fix this syntax for the rest of the paper, so that ‘predicate’ refers to predicates that can mention elements of $\bigcup_i V_i$ as constants.

Remark 4.6. There is an apparent mismatch between Definitions 4.3 and 2.1: variable levels in Definition 2.1 start from 1 whereas the hierarchy in Definition 4.3 starts at 0. This is purely a technical matter: the sequence of universes $V_0, V_1, V_2, \ldots$ only attains its full size of $\beth(\omega+1)$ from level 1; see Theorem 5.1.
Suppose that \( \exists \) holds. Then suppose \( \forall \). The rest follows by routine arguments unpacking the definitions.

We work by induction on \( \text{level}(a) \).

Lemma 4.11 (Extensionality). Suppose \( i \geq 1 \) and \( x, y \in V_i \). Then

\[
\vdash x = y \iff (\forall c. c \in x \iff c \in y).
\]

Proof. The key observation is that by construction \( x \) is a set, and \( y \) is a set, and sets are extensional. The rest follows by routine arguments unpacking the definitions. \( \square \)

4.1.2. Closure properties. Recall \( E \land E' \) from Definition 3.11:

Lemma 4.12. Suppose \( i \geq 0 \). Then

\( E, E' \in \text{EQ}_i \) implies \( E \land E' \in \text{EQ}_i \).

Proof. We work by induction on \( i \):

1. Suppose \( i = 0 \). Then \( E, E' \in \text{EQ}_0 \) when \( E \) and \( E' \) are equivalence relations on \( V_0 \) and \( \#E, \#E' < \exists(\omega) \). By Lemma 3.15(3) \( \#(E \land E') < \exists(\omega) \), so we are done.

2. Suppose \( i \geq 1 \). The cardinality argument is as before. Fix some \( 0 \leq i' < i \). By assumption there exist \( F, F' \in \text{EQ}_{i'} \) with \( F \$ E \) and \( F' \$ E' \). By inductive hypothesis \( F \land F' \in \text{EQ}_{i'} \) and by Lemma 3.21 \( F \land F' \$ E \) and \( F \land F' \$ E' \). It is routine to check that \( (F \land F') \$ (E \land E') \). \( \square \)

Proposition 4.13. Suppose \( i \geq 0 \). Then:

1. Suppose \( X \in V_{i+1} \). Then \( \bigcap X \in V_i \).

In words: the intersection of a very simple collection of sets, is very simple.
(2) If \( X \in \mathcal{V}_i \) then \( \mathcal{V}_i \setminus X \in \mathcal{V}_i \).

In words: the complement of a very simple set, is very simple.

**Proof.** (1) By construction each \( X \in \mathcal{X} \) is a subset of \( \mathcal{V}_{i-1} \), thus \( \mathcal{X} \subseteq \mathcal{V}_{i-1} \).

Now suppose \( E \in \text{EQ} \) for \( i' \leq i-2 \) and \( E \mathcal{X} \) (so \( E \) is a very simple equivalence relation on \( \mathcal{V}_{i'} \) and it supports \( \mathcal{X} \)). It is then a fact of sets that \( ES \mathcal{X} \).

(2) By simple calculations exploiting the fact that permutations are bijective. \( \square \)

### 4.1.3. Support at every level

**Lemma 4.14.** Suppose \( i, i' \geq 0 \) and suppose \( X \in \mathcal{V}_i \) and \( E \in \text{EQ}_i \). Then:

- There exists \( F \in \text{EQ}_{i'} \) such that \( F \mathcal{X} \) (even if \( i' \geq i-1 \)).
- There exists \( F \in \text{EQ}_{i'} \) such that \( F \mathcal{X} \) (even if \( i' \geq i \)).

**Proof.** — If \( i' < i-1 \) then this is by assumption in Definition 4.3 above.

— If \( i' = i-1 \) then we note that \( X \) is supported by \( F = \{ X, \mathcal{V}_{i-2} \setminus X \} \).

— If \( i' = i \) then we note that \( X \) is supported by \( F = \{ X, \mathcal{V}_{i-1} \setminus X \} \).

— If \( i' > i \) then the permutation action is trivial (Definition 3.23) and \( X \) is supported by \( F = \{ \mathcal{V}_{i-1} \} \).

The case of \( E \) is similar; if \( i' = i \) then \( E \) is supported by itself. \( \square \)

**Remark 4.15** (A word on terminology). We do not just call our hierarchy in Definition 4.3 ‘hereditarily simple’ because the condition seems a little stronger than that. Not only must \( X \in \mathcal{V}_{i+1} \) consist of very simple elements, it must also be supported over every lower level by some small equivalence relation which itself must be very simple. ‘Hereditarily simple’ would suggest, to this author, just being supported at the single level below.

**Remark 4.16.** Definition 4.3 asserts the existence of supporting \( E \), but does not pin down what they should be. A stronger condition is also plausible: given a set \( X \) with a permutation action over \( \mathcal{A} \), an element \( x \in X \) induces an equivalence \( \text{symm}(x) \) on \( \mathcal{A} \) such that \( (a, a') \in \text{symm}(x) \) when \( (a' a) \cdot x = x \). Then we could replace the ‘there exists an equivalence relation’ with ‘is supported by \( \text{symm} \)’ throughout in Definition 4.3. This stronger condition in essence excludes the possibility of fuzzy support as discussed in [Gab07, Subsection 6.2].

### 4.2. Soundness of comprehension

**Definition 4.17** (1) Suppose \( i \geq 1 \) and \( a \in \text{Var}_i \). If \( \phi \) has (at most) one free variable \( a \) and is in quantifier normal form (so \( \phi \) has the form \( Q \phi' \) where \( \phi' \) is quantifier-free and \( Q \) represents a \( \forall \exists \)-quantifier prefix; \( \phi \) can mention constants), then call \( \phi \) a-**comprehensive**.

(2) Define a mapping \( \Gamma a. \phi \) from \( a \)-comprehensive predicates \( \phi \) to elements \( \Gamma a. \phi \in \bigcup_n \text{pow}^n (\mathcal{V}_i) \) as follows:

\[
\begin{align*}
\Gamma a. \phi &= \{ x \in \mathcal{V}_i \mid \models \phi[a:=x] \} & \phi \text{ is quantifier-free} \\
\Gamma a. (\forall b. \phi) &= \{ \Gamma a. (\phi[b:=y]) \mid y \in \mathcal{V}_j \} & b \in \text{Var}_j \\
\Gamma a. (\exists b. \phi) &= \{ \Gamma a. (\phi[b:=y]) \mid y \in \mathcal{V}_j \} & b \in \text{Var}_j
\end{align*}
\]

(So \( \forall b. \phi \) and \( \exists b. \phi \) map to the same set.)

**Lemma 4.18.** Suppose \( i \geq 1 \) and \( a \in \text{Var}_i \) and \( \phi \) is \( a \)-comprehensive with a quantifier prefix of length \( n \geq 0 \). Then

\[ \Gamma a. \phi \in \mathcal{V}_{i+n+1}. \]

**Proof.** We reason by induction on the length of the quantifier prefix of \( \phi \):

(1) \( \text{Suppose} \ n = 0, \ \text{so} \ \phi \ \text{is quantifier-free}. \)

By construction \( \Gamma a. \phi \subseteq \mathcal{V}_i \).

Now suppose \( \text{consts} \left( \phi \right) = \{ y_1, \ldots, y_n \} \). By Lemmas 4.14, 3.22, and 4.12 for each \( 0 \leq i' \leq (i+1)-2 \) there exists \( F_{i'} \in \text{EQ}_{i'} \) such that \( F_{i'} \mathcal{S}y_1, \ldots, y_n \).
Choose \(0 \leq i' \leq (i+1)-2\) and consider some \(\pi \in \text{stab}(F_{i'})\). We note that \(\pi \cdot y_1 = y_1, \ldots, \pi \cdot y_n = y_n\) and therefore for each \(y \in \{y_1, \ldots, y_n\}\) of an appropriate level,
\[
\pi \cdot x \in y \iff x \in y \quad \text{and} \quad y \in \pi \cdot x \iff y \in x \quad \text{and} \quad x = y \iff x = \pi \cdot y.
\]

Since \(\pi\) was arbitrary, it follows by a routine calculation that \(F_{i'} \Pi \pi a. \phi = \{x \in V_i \mid \models \phi[a_1 = x]\}\).

It follows that \(\Gamma a. \phi \in V_{i+1}\).

(2) Suppose \(n \geq 1\).

As before suppose \(\text{consts}(\phi) = \{y_1, \ldots, y_n\}\), and by Lemmas 4.14, 3.22, and 4.12 for each \(0 \leq i' \leq (i+n+1)-2\) there exists \(F_{i'} \in \text{EQ}_{i'}\) such that \(F_{i'} \Pi y_1, \ldots, y_n\).

Using the inductive hypothesis we have that \(\Gamma a. \phi \subseteq V_{i+n}\), and by a routine calculation using the fact that permutations are bijective, \(F_{i'} \Pi \Gamma a. \phi\) for every \(0 \leq i' \leq (i+n+1)-2\), so that \(\Gamma a. \phi \in V_{i+n+1}\) as required.

**Proposition 4.19.** Suppose \(\phi\) is a predicate with constants \(\{y_1, \ldots, y_n\}\) and one free variable \(a \in \text{Var}_i\) where \(i \geq 1\). Then
\[
\{x \in V_i \mid \models \phi\} \in V_{i+1}.
\]

**Proof.** Assume without loss of generality that \(\phi\) is in quantifier normal form, so that \(\phi\) is \(a\)-comprehensive (Definition 4.17). By Lemma 4.18 \(\Gamma a. \phi \in V_{i+n+1}\) where \(n\) is the length of the quantifier prefix of \(\phi\) when written in quantifier normal form (this can always be done).

We can recover \(\{x \in V_i \mid \models \phi\}\) from \(\Gamma a. \phi\) using a sequence of intersections \(\bigcap\) and unions \(\bigcup\), according as the quantifier prefix contains \(\forall\) and \(\exists\). By Proposition 4.13, \(\{x \in V_i \mid \models \phi\} \in V_{i+1}\) \(\Box\)

### 5. Counting Elements and Support

#### 5.1. Nonzero levels have the same size

Recall from Definition 4.3 the hierarchy of very simple sets. We notice that the support restrictions in that Definition keep cardinality under strict control:

**Theorem 5.1(1)** \(\#V_0 = \bigcup(\omega)\), and \(\#V_i = \bigcup(\omega+1)\) for every \(i > 0\).

**Theorem 5.1(2)** \(\#\text{EQ}_i = \bigcup(\omega+1)\) for every \(i \geq 1\).

**Proof.** Before giving details we suggest why this result is plausible: Definition 4.3 is set up so that every \(X \in V_i\) for \(i \geq 2\) is a subset of \(V_{i-1}\) that is symmetric modulo some equivalence relation over \(V_{i-2}\) with fewer than \(\bigcup(\omega)\) elements (in our terminology: it is small).

Therefore, our only choices when building \(X\) modulo this symmetry are the cardinalities of the intersection of \(X\) with each equivalence class. Our assumption of GCH in Definition 3.8 limits these cardinalities to \(\omega\) many possible values.

It follows that the number of \(X \in V_i\) is dominated by the number of small equivalence relations over \(V_{i-2}\). It is plausible (and we shall now prove) that this will prevent size from increasing as we move up the hierarchy, beyond level 1. Now for the details:

1. From Definitions 4.3(1) and 4.1, \(\#V_0 = \bigcup(\omega)\).
2. From Definition 4.3(3) an \(x \in V_1\) is just a set of atoms. It is a fact of sets that \(\#V_1 = \bigcup(\omega+1)\).
3. From Definition 4.3(2) an equivalence relation \(E \in \text{EQ}_0\) is just a small partition on \(\bigcup\). It is a fact of sets that \(\#\text{EQ}_0 = \bigcup(\omega+1)\).
4. Now consider \(X \in V_2\), so \(X \subseteq V_1\) is a set of sets of atoms. By condition 3 of Definition 4.3 there exists a very simple \(E_0 \in \text{EQ}_0 = \text{Rel}(\bigcup)\) with \(E_0 X\).

We noted above that there are \(\bigcup(\omega+1)\) many possible \(E_0\), so fix one particular \(E_0\). If we can show that for each possible \(E_0\) there are at most \(\bigcup(\omega+1)\) many possible \(X\) that it could support, then we will be done – since it is a fact of cardinal arithmetic that \(\bigcup(\omega+1) \times \bigcup(\omega+1) = \bigcup(\omega+1)\).

So consider some possible \(X\) such that \(E_0 X\). An element \(x \in X\) has the form \(x \in V_1\) and is just a set of atoms, and because \(E_0 X\) we know that \(x \in X \iff \pi \cdot x \in X\) for every \(\pi \in \text{stab}(E_0)\) (Definition 3.20).
We see that modulo the action of permutations in $\text{stab}(E_0)$, $x \in V_1$ can be represented by the mapping taking each equivalence class $e \in E_0$ to the cardinality $\#(x \cap e)$ (that is, $e$ maps to the number of atoms from $e$ that are contained in $x$). By Lemma 3.7 there are at most $\omega$ many such cardinalities, because we assumed GCH in Definition 3.8.

By assumption $\#E_0 < \mathfrak{A}(\omega)$, so (for fixed $E_0$) there are fewer than $\mathfrak{A}(\omega)$ many possible values for each $x$ (up to $\text{stab}(E_0)$).

Thus there are no more than $\mathfrak{A}(\omega+1)$ many possibilities for $X$.

(5) Consider $E \in EQ_1$, so $E \subseteq V_1 \times V_1$ is a relation on sets of atoms, that is, a set of pairs of sets of atoms. The reasoning below is essentially identical to that for $X \in V_2$ (where $X$ is a set of sets of atoms), but we give full details anyway.

By assumption there exists a very simple $E_0 \in EQ_0 \subseteq \text{Rel}(\mathcal{A})$ such that $E_0 \$ E$.

There are $\mathfrak{A}(\omega+1)$ many possible $E_0$, so fix one particular $E_0$. If we can show that there for each possible $E_0$ there are at most $\mathfrak{A}(\omega+1)$ many possible $E$ that it could support, then we will be done.

So consider some possible $E$. An element of $E$ is a pair $(x, x') \in V_1 \times V_1$ and is just a pair of sets of atoms, and $(x, x') \in E \iff (\pi_x, \pi_{x'}) \in E$ for every $\pi \in \text{stab}(E_0)$.

We see that up to $\text{stab}(E_0)$, $(x, x')$ can be represented by the mapping taking each equivalence class $e \in E_0$ to a tuple of cardinalities $(\#(x \cap e), \#(x' \cap e), \#(x \cap x' \cap e), \#((x \setminus x') \cap e), \#((x' \setminus x) \cap e))$.

By assumption $\#E_0 < \mathfrak{A}(\omega)$, so (for fixed $E_0$, and using our assumption of GCH) there are fewer than $\mathfrak{A}(\omega)$ many possible values for each pair $(x, x')$ (up to $\text{stab}(E_0)$).

Thus there are no more than $\mathfrak{A}(\omega+1)$ many possibilities for $E$.

(6) Now consider $X \in V_3$. By assumption there exist $E_0 \in EQ_0$ (relating atoms $a \in V_0$) and $E_1 \in EQ_1$ (relating sets of atoms $x \in V_1$) such that $E_0 \$ X$ and $E_1 \$ X$.

We have proved above that there are at most $\mathfrak{A}(\omega+1)$ many possible values for $E_1$. We now count the possibilities for $X$, given a fixed choice of $E_1$.

By assumption, $\#E_1 < \mathfrak{A}(\omega)$. If $\pi \in \text{stab}(E_1)$ and $X \in V_2$ then $X \in X \in V_3$ if and only if $\pi \cdot X \in X$.

Thus $X$ can be represented as a mapping from the $< \mathfrak{A}(\omega)$ many equivalence classes $e \in E_1$, to the cardinality $\#(X \cap e)$. So (for each fixed $E_1$) there are no more than $\mathfrak{A}(\omega)$ many possibilities for $X$ and so no more than $\mathfrak{A}(\omega+1)$ many possibilities for $X$.

We proved above that there are at most $\mathfrak{A}(\omega+1)$ many possible $E_1$ and it follows by cardinal arithmetic that there are at most $\mathfrak{A}(\omega+1)$ many possible values for $X$.

(7) The case of $E \in EQ_2$ is exactly similar to the case of $X \in V_3$.

(8) The case of $X \in V_i$ for $i \geq 4$ is identical to the case of $X \in V_3$; we just note that there are $\mathfrak{A}(\omega+1)$ many possible $E_{i-2} \in EQ_{i-2}$, and for each $E_{i-2}$ there are $< \mathfrak{A}(\omega)$ many possibilities.

(9) The cases of $E \in EQ_i$ for $i \geq 3$ are exactly similar to the corresponding cases for $X \in V_{i+1}$.

Remark 5.2. Theorem 5.1 might look like it gives us Typical Ambiguity immediately, just by choosing any bijection $\Theta : V_1 \cong V_2$ and propagating it in the natural pointwise manner up the hierarchy of sets, so if $X \in V_i$ for $i \geq 2$ then we just take $\Theta(X) = \{\Theta(x) \mid x \in X\}$.

This does not work because we do not know that if $X$ is very simple (Definition 4.3) then so is $\Theta(X)$, for $i \geq 2$. If we think of elements $X \in V_i$ as being subject to quite strict symmetry conditions, then the problem with applying Theorem 5.1 directly is that the bijection $\Theta : V_1 \cong V_2$ may be highly asymmetric and may destroy these conditions when applied pointwise to sets.

So we need to be more careful about how we use the isomorphisms that Theorem 5.1 gives us. We do this next.

5.2. A closer analysis of levels 1 and 2

We continue Remark 5.2. Theorem 5.1 has a relatively long proof but it is essentially an elementary observation.

---

4 We are not trying to be elegant here: we just need to provide enough information to reconstruct $E$. 

In this Subsection we undertake a nonelementary analysis of the fine structure of our universe, laying the groundwork for the ‘squashing properties’ which will follow in Subsection 5.3. So in a certain sense, this is where things start to get interesting.

5.2.1. An equivariant large partition. We gather together some definitions and notations which will be useful henceforth. Recall from Definition 4.3 and Remark 4.4 that \( V_1 \) is just the set of subsets of \( V_0 = A \).

**Definition 5.3.** Let \( T = \{ T_k \subseteq V_1 \mid k < \omega \} \) be a set of subsets of \( V_1 \) such that:

1. The \( T_k \) form an increasing sequence of subsets of \( V_1 \):
   \[
   T_0 \subset T_1 \subset T_2 \subset \cdots \subset V_1
   \]
2. For every \( k < \omega \) we have
   \[
   \#T_k = \mathfrak{A}(\omega+1) = \#(T_{k+1} \setminus T_k)
   \]
   and also
   \[
   \#(V_1 \setminus \bigcup_{k<\omega} T_k) = \mathfrak{A}(\omega+1).
   \] (1)
3. Each \( T_k \) is equivariant over \( V_0 \), meaning that:
   \[
   \forall \pi \in \text{Perm}(V_0). \pi \cdot T_k = T_k.
   \]
   (This is the pointwise action from Definition 3.19, so \( \pi \cdot T_k = \{ \pi(a) \mid a \in x \mid x \in T_k \} \).)

We call the \( T_k \) slices of \( V_1 \).

Furthermore, we define

\[
S_k = \{ \{ x \} \mid x \in T_k \} \subseteq V_2
\]
and write \( S = \{ \{ x \} \mid x \in V_1 \} \), so that

\[
S_0 \subset S_1 \subset S_2 \subset \cdots \subset \bigcup_{k<\omega} S_k \subset S \subset V_2.
\]

Note that it follows from condition 3 above that \( S_k \) is also equivariant over \( V_0 \), meaning that:

\[
\forall \pi \in \text{Perm}(V_0). \pi \cdot S_k = S_k.
\]

**Remark 5.4.**

1. In Definition 5.3 we have partitioned \( V_1 \) into a sequence of large\(^5\) slices.
2. Incrementing \( k \) grows each slice by a large amount.
3. The union \( \bigcup_{k<\omega} T_k \) does not exhaust \( V_1 \); a large slice of elements \( V_1 \setminus \bigcup_{k<\omega} T_k \) remains which is in bijection all of the following sets (see Lemma 5.6):
   \[
   V_1, \quad \bigcup_{k<\omega} T_k \subseteq V_1, \quad \text{and} \quad V_2 \setminus \bigcup_{k<\omega} S_k \subseteq V_2.
   \]
   This is deliberate and is needed for Propositions 5.24 and 5.25.
4. Condition 3 of Definition 5.3 is needed for Lemmas 6.18 and 6.21.
5. We need a name for this partition, so we will call this an equivariant large partition of \( V_1 \).

**Lemma 5.5.** An equivariant large partition exists, as described in Definition 5.3 and Remark 5.4.

**Proof.** It suffices to exhibit one.

— Let \( T_0 \) consist of all subsets of \( V_0 \) that have cardinality divisible by 2, and 3.
— Let \( T_1 \) consist of all subsets of \( V_0 \) that have cardinality divisible by 2, and by 3 or 5.
— Let \( T_2 \) consist of all subsets of \( V_0 \) that have cardinality divisible by 2, and by 3, 5, or 7.
— Let \( T_3 \) consist of all subsets of \( V_0 \) that have cardinality divisible by 2, and by 3, 5, 7, or 11.

\(^5\)Following Notation 3.12 and Theorem 5.1, ‘large’ means ‘bijets with \( V_1 \)’ and ‘has cardinality \( \#\mathfrak{A}(\omega+1) \)’.
— Let $T_3$ consist of all subsets of $V_0$ that have cardinality divisible by 2, and by 3, 5, 7, 11, or 13.
— In general, we take $T_i$ for $i < \omega$ to be subsets that have cardinality divisible by 2 and by at least one of the first $i$ primes greater than 2.

It is routine to verify that this suffices. □

Lemma 5.6 gives a helpful overview:

**Lemma 5.6.** The following sets all have cardinality $\beth(\omega+1)$ (and so can be bijected):

1. $T_k$ and $S_k$ for every $k < \omega$.
2. $T_k' \setminus T_k$ and $S_k' \setminus S_k$ for any $k < k' < \omega$.
3. $V_1 \setminus \bigcup_{k<\omega} T_k$.
4. $V_2 \setminus S$.
5. $V_2 \setminus \bigcup_{k<\omega} S_k$.
6. $V_1$, $V_2$, and any other $V_i$ for $i \geq 1$.

**Proof.**— Parts 1, 2, and 3 are direct from Definition 5.3.
— $\#(V_2 \setminus S) = \beth(\omega+1)$ is by an easy counting argument using Theorem 5.1.
— $\#(V_2 \setminus \bigcup_{k<\omega} S_k) = \beth(\omega+1)$ follows from parts 3 and 4.
— Part 6 just repeats Theorem 5.1. □

**Remark 5.7** (Permissive nominal sets). The idea of a nominal notion of support based on a partition of atoms into two halves of equal cardinality, which is what the slices $T_k$ and $S_k$ are all about, goes back to the *permission sets* of papers like [DGM10]—contrast this with the nominal techniques from [GP01], based on splitting atoms into halves where one half is small (meaning finite) and the other is large (meaning countable).

Here the idea is taken further: technicalities aside, we have not one partition but an infinite ascending sequence of them. Why? We will need this infinite ascending sequence to handle quantifiers: we see this most clearly perhaps in the last two lines of Figure 8, for $\exists a. \phi$, where going under a quantifier we move from level $k$ to level $k+1$.

**5.2.2. Support with respect to the partition**

**Remark 5.8.** Recall the notation $S\%X$ from Definition 3.24. So
— for $i \geq 1$ we can write $T_k\%X$ for $X \subseteq V_i$ and $T_k\%E$ for $E \subseteq V_i \times V_i$, and
— for $i \geq 2$ we can write $S_k\%X$ for $X \subseteq V_i$ and $S_k\%E$ for $E \subseteq V_i \times V_i$,

since $V_i$ has a pointwise permutation action over $T_k$ (if $i \geq 1$) and over $S_k$ (if $i \geq 2$), as outlined in Definition 3.19.

Note that we do not need to know that $X \in V_{i+1}$ to sensibly write $S_k\%X$; we just need to know $X \subseteq V_i$.

**Notation 5.9.** Continuing Remark 5.8, we extend the notation $T_k\%(-)$ and $S_k\%(-)$ to $T_\exists\%(-)$ and $S_\exists\%(-)$, by:

$$T_\exists\%(-) \quad \text{when} \quad \exists k < \omega. T_k\%(-)$$
$$S_\exists\%(-) \quad \text{when} \quad \exists k < \omega. S_k\%(-)$$

**Remark 5.10.** Note that $T_\exists\%(-)$ when a finite $k$ exists (we do not permit $k = \omega$), and similarly for $S_\exists\%(-)$. In general, we will be most interested in $T_k$ and $S_k$ for $k < \omega$ henceforth.

**Definition 5.11.** Define sets $V_{i,\exists 1} \subseteq V_i$ and $EQ_{i,\exists 1} \subseteq EQ_i$ for $i \geq 1$ and $V_{i,\exists 2} \subseteq V_i$ and $EQ_{i,\exists 2} \subseteq EQ_i$ for $i \geq 2$ as in Figure 5.

**Definition 5.12.** Suppose $i \geq 1$ and $\pi \in Perm(V_i)$ and $k < \omega$. Recall from Definition 3.24 that:
— $T_k\%\pi$ when $T_k\%x$ for every $x \in nontriv(\pi)$, and
— if $i \geq 2$ then $S_k\%\pi$ when $S_k\%x$ for every $x \in nontriv(\pi)$, and
— if $i = 1$ then $T_k \$ \pi$ when $\text{nontriv}(\pi) \subseteq T_k$, and
— if $i = 2$ then $S_k \$ \pi$ when $\text{nontriv}(\pi) \subseteq S_k$.

Following Notation 5.9 we define $T_3$ and $S_3 \$ \pi$ by

$$
T_3 \$ \pi \quad \text{when } \exists k < \omega. T_k \$ \pi \quad \text{and}
S_3 \$ \pi \quad \text{when } \exists k < \omega. S_k \$ \pi.
$$

Remark 5.13. If the essence of nominal techniques is to consider sets acted on by some symmetry group and the action satisfies some kind of support property—Subsection 3.4 and Definition 3.20 in this paper contain examples—then an element $X \in V_{i,32}$ is nominal in two ways:

— $X \in V_{i,32}$ is acted on by $\text{Perm}(V_j)$ and supported by some $E_j \in \text{EQ}_j$ for $0 \leq j \leq i-2$, in the sense that if $\pi \in \text{stab}(E_j)$ (so that $\pi \in V_j$ and $\pi$ respects $E_j$-equivalence classes) then $\pi \cdot X = X$.

— $X \in V_{i,32}$ is in addition supported by $S_k \subseteq V_2$, in the sense that $\pi \in \text{fix}(S_k)$ (so $\pi \in \text{Perm}(V_2)$ and $\pi(a) = a$ for every $a \in S_k \subseteq V_2$) implies $\pi \cdot X = X$.

These two notions of support interact in the ‘squashing’ property, Theorem 5.20.

**Lemma 5.14.** Suppose $X \subseteq V_1$. Then $X \in V_{2,32}$ if and only if there exist $k < \omega$ and $x \in T_k$ such that $X = \{x\}$.

**Proof.** From Definitions 5.11 and 3.24.

**Lemma 5.15.** (1) Suppose $\pi \in \text{Perm}(V_1)$ and $T_3 \$ \pi$. Suppose $i \geq 1$ and $x \in V_i$. Then

$$
T_3 \$ x \quad \text{if and only if } \quad T_3 \$ \pi \cdot x.
$$

(2) Suppose $\pi \in \text{Perm}(V_2)$ and $S_3 \$ \pi$. Suppose $i \geq 2$ and $x \in V_i$. Then

$$
S_3 \$ x \quad \text{if and only if } \quad S_3 \$ \pi \cdot x.
$$

**Proof.** By a routine calculation using Lemma 3.25.\(^\text{6}\)

**Remark 5.16.** Lemma 5.15 does not hold for general $\pi$. But if the support of $\text{nontriv}(\pi)$ is bounded above by some finite $k$ then it is plausible that the support of $x$ is bounded above if and only if the support of $\pi \cdot x$ is bounded above.

**Lemma 5.17.** is a familiar technical ‘nominal’ property:

**Lemma 5.17.** (1) Suppose $i \geq 1$ and $x \in V_i$ and $X \in V_{i+1}$ and $\pi \in \text{fix}(T_k)$.\(^\text{7}\) Suppose $k < \omega$ and $T_k \$ X$. Then

$$
x \in X \quad \text{if and only if } \quad \pi \cdot x \in X.
$$

(2) Suppose $i \geq 2$ and $x \in V_i$ and $X \in V_{i+1}$ and $\pi \in \text{fix}(S_k)$. Suppose $k < \omega$ and $S_k \$ X$. Then

$$
x \in X \quad \text{if and only if } \quad \pi \cdot x \in X.
$$

**Proof.** By Lemma 3.27(1) (since $S_k \$ X, X’) $\pi \cdot X = X$. By the construction of the pointwise action on $X$ (Definition 3.19), $x \in X$ if and only if $\pi \cdot x \in \pi \cdot X$. The result follows.\(^\text{8}\)

\(^{6}\)The calculation is routine but only given the assumptions that $T_3 \$ \pi$ and $S_3 \$ \pi$: without such an assumption $\pi \cdot x$ might be too ‘spread out’.

\(^{7}\)So $\pi \in \text{Perm}(V_1)$ and $\text{nontriv}(\pi) \cap T_k = \varnothing$.
5.3. Squashing properties
In this Subsection we will prove some squashing properties:

— Theorem 5.20.
— Its immediate corollary Corollary 5.21.
— Its eventual corollaries Propositions 5.24 and 5.25.
— Its most technical corollary Proposition 5.28, which we discuss in detail in Remark 5.27.

These all describe various senses, which are made formal, in which everything that happens in \( V_i \) can be sufficiently represented in \( V_{i,\geq 2} \) or \( V_{i,\geq 1} \) up to the action of a suitable permutation, which we call a squashing permutation.

5.3.1. Squashing elements. We prove Theorem 5.20, which gives a general sense in which we may ‘squash’ the support of an element.

Lemma 5.18 is a key technical observation:

**Lemma 5.18.** Suppose \( k < \omega \). Then:

1. If \( F \in EQ_1 \) then there exists \( \tau \in \text{fix}(T_k) \) such that \( T_{k+1}\uparrow\tau\cdot F \).
2. If \( F \in EQ_2 \) then there exists \( \tau \in \text{fix}(S_k) \) such that \( S_{k+1}\uparrow\tau\cdot F \).

**Proof.** We consider part 2; part 1 is exactly similar.

By construction in Definition 5.3 \( S_{k+1} \setminus S_k \subseteq V_2 \) is large (has cardinality \( \aleph(\omega+1) \)). By construction in Definition 4.3(4) \( F \) is a small set of equivalence classes on \( V_2 \), each of which (by Theorem 5.1) has cardinality at most \( \aleph(\omega+1) \).

It follows using Lemma 3.28 and some elementary cardinality arguments that a permutation \( \tau \in \text{fix}(S_k) \) exists such that \( S_{k+1}\uparrow\tau\cdot F \); it suffices to choose some equivalence class \( f \in F \) such that \( f \setminus S_k \) is large, and then let \( \tau \) ‘squash’ the intersection of any other equivalence classes with \( V_2 \setminus S_k \) down into \( S_{k+1} \setminus S_k \), so that \( V_2 \setminus S_{k+1} \subseteq \tau\cdot f \).

**Lemma 5.19.** Suppose \( k < \omega \). Then:

1. If \( F \in EQ_1 \) and \( i \geq 1 \) and \( x \in V_i \) then
   \[ T_k\uparrow F \land F\uparrow x \quad \text{implies} \quad T_k\uparrow x. \]
2. If \( F \in EQ_2 \) and \( i \geq 2 \) and \( x \in V_i \) then
   \[ S_k\uparrow F \land F\uparrow x \quad \text{implies} \quad S_k\uparrow x. \]

**Proof.** We consider just part 2. Unpacking Definition 3.24(2&3), \( S_k\uparrow x \) means that if \( \pi \in \text{fix}(S_k) \) then \( \pi\cdot x = x \), and \( F\uparrow x \) means that if \( \pi \in \text{stab}(F) \) then \( \pi\cdot x = x \).

We now note of Definition 3.24(3) that if \( S_k\uparrow F \) then \( V_2 \setminus S_k \) must be contained in a single equivalence class of \( F \), and it follows that if \( \pi \in \text{fix}(S_k) \) then \( \pi \in \text{stab}(F) \).

Theorem 5.20 has a short and fairly simple proof, but as mentioned in Remark 5.13 it is a key property where our two notions of symmetry (\( E\uparrow X \), and \( S_k\uparrow X \) or \( T_k\uparrow X \)) interact:

**Theorem 5.20.** (The first squashing property) Suppose \( k < \omega \). Then:

1. If \( i \geq 1 \) and \( x \in V_i \) then there exists \( \tau \in \text{fix}(T_k) \) (so \( \tau \in \text{Perm}(V_1) \) and nontriv(\( \pi \)) \( \cap T_k = \emptyset \)) such that \( T_{k+1}\uparrow\tau\cdot x \).
2. If \( i \geq 2 \) and \( x \in V_i \) then there exists \( \tau \in \text{fix}(S_k) \) (so \( \tau \in \text{Perm}(V_1) \) and nontriv(\( \pi \)) \( \cap T_k = \emptyset \)) such that \( S_{k+1}\uparrow\tau\cdot x \).

**Proof.** We consider just part 2; part 1 is exactly similar. By Lemma 4.14 (since \( x \in V_i \)) there exists \( E \in EQ_2 \) such that \( E\uparrow x \). By Lemma 5.18 there exists \( \tau \in \text{fix}(S_k) \) such that \( S_{k+1}\uparrow\tau\cdot E \), and by Lemma 3.25 \( \tau\cdot E\uparrow \tau\cdot x \). By Lemma 5.19, \( S_{k+1}\uparrow\tau\cdot x \) as required.

Corollary 5.21 is an attractive corollary of the squashing property, which will be useful later:
Corollary 5.21(i) Suppose \( i \geq 1 \) and \( X, X' \subseteq V_i \) and suppose there exists \( k < \omega \) such that \( T_k \S X, X' \) (in particular it would suffice that \( X, X' \in V_{i+1, \omega} \)). Then

\[
\forall x \in V_{i, \omega} \cdot (x \in X \leftrightarrow x \in X') \quad \text{implies} \quad X = X'.
\]

As a corollary, if \( X \neq X' \) then there exists \( x \in V_{i, \omega} \) (not just \( x \in V_i \)) such that \( x \in X \) and \( x \notin X' \), or \( x \notin X \) and \( x \in X' \).

(2) Suppose \( i \geq 2 \) and \( X, X' \subseteq V_i \) and suppose there exists \( k < \omega \) such that \( S_k \S X, X' \) (in particular it would suffice that \( X, X' \in V_{i+1, \omega} \)). Then

\[
\forall x \in V_{i, \omega} \cdot (x \in X \leftrightarrow x \in X') \quad \text{implies} \quad X = X'.
\]

As a corollary, if \( X \neq X' \) then there exists \( x \in V_{i, \omega} \) such that \( x \in X \) and \( x \notin X' \), or \( x \notin X \) and \( x \in X' \).

Proof. We prove part 1; part 2 is exactly similar.

Suppose of each \( x \in V_{i, \omega} \) that \( x \in X \) if and only if \( x \in X' \).

Now consider \( x \in X \) such that \( T_3 \S x \) does not necessarily hold; if we can prove \( x \in X \leftrightarrow x \in X' \) then we will have \( X = X' \) as required, by extensionality of sets.

By Theorem 5.20 we can find a \( \pi \in \fix(T_k) \) such that \( T_{k+1} \S (\pi \cdot x) \). We reason as follows:

\[
\begin{align*}
x \in X & \iff \pi \cdot x \in X & T_k \S X, \pi \in \fix(T_k) & \text{Lemma 3.27} \\
& \iff \pi \cdot x \in X' & \text{Assumption} \\
& \iff x \in X' & T_k \S X', \pi \in \fix(T_k) & \text{Lemma 3.27}
\end{align*}
\]

5.3.2. Squashing permutations. Propositions 5.24 and 5.25 give a sense in which we may squash permutations.

Lemma 5.22. Suppose \( i \geq 2 \) and \( x \in V_i \) and \( k < \omega \) and \( S_k \S x \) and \( \neg(S_k \S x) \). Then

\[
\# \{ \tau \cdot x \mid \tau \in \fix(S_k), \; S_k \S \tau \cdot x, \; \neg(S_k \S \tau \cdot x) \} = \Delta(\omega+1).
\]

Proof. Write \( L = S_{k+1} \setminus S_k \). By assumption in Definition 5.3 \( L \) is large, so we can partition it into a large set of large subsets \( \{ L_l \mid l < \Delta(\omega+1) \} \). So \( \# L_l = \Delta(\omega+1) \) and if \( l_1 \neq l_2 \) then \( L_{l_1} \) and \( L_{l_2} \) are disjoint.

For each \( l \) choose some permutation \( \tau_l \in \fix(S_k) \) such that \( \tau_l \cdot L = L_l \). It is a fact that we can do this; we will prove that the map \( l \mapsto \tau_l \cdot x \) is injective.

Suppose it is not, so that we have \( l_1, l_2 < \Delta(\omega+1) \) such that \( \tau_{l_1} \cdot x = \tau_{l_2} \cdot x \); write \( y \) for this common value. By Lemma 3.25

\[
\tau_{l_1} \cdot S_{k+1} = L_{l_1} \quad \text{and} \quad \tau_{l_2} \cdot S_{k+1} = L_{l_2} \quad \text{and} \quad L_{l_1}, L_{l_2} \S y.
\]

By construction \( L_{l_1} \cap L_{l_2} = S_k \) and by Lemma 3.26 \( S_k \S y \). It follows by Lemma 3.25 again that \( S_k \S x \), a contradiction.

Lemma 5.23. Suppose that:

\( i \geq 3 \) and \( X \in V_i \) and \( 2 \leq j \leq i-1 \) and \( E \in \text{EQ}_j \) and \( E \S X \) and \( k < \omega \) and \( S_k \S E \).

\( y'', y \in V_j \) and \( (y'', y) \in E \) and \( S_k \S y'' \) and \( \neg(S_k \S y) \).

Then there exist a large number of \( y' \in V_j \) such that:

\( \neg S_{k+1} \S y' \) and \( \neg(S_k \S y') \).

\( (y'', y') \in E \).

\( y' = \tau \cdot y \) for some \( \tau \in \fix(S_k) \).

In symbols:

\[
\# \{ \tau \cdot y \in V_j \mid S_{k+1} \S \tau \cdot y, (y'', \tau \cdot y) \in E, \tau \in \fix(S_k) \} = \Delta(\omega+1).
\]
Proof. By Theorem 5.20 there exists a $\tau \in \text{fix}(S_k)$ such that $S_{k+1}^\tau \cdot y$, and by Lemma 3.25 (since $\tau \in \text{fix}(S_k)$) also $\neg(S_k \tau \cdot y)$. The result follows from Lemma 5.22.

PROPOSITION 5.24 (The second squashing property: level 2). Suppose $i \geq 3$ and $2 \leq j \leq i-1$ and $E \in \text{EQ}_{j, \geq 2}$ and $k < \omega$ and $S_k \varepsilon E$. Suppose $\pi \in \text{stab}(E)$, so that $\pi \in \text{Perm}(V_j)$.

Then there exists $\pi' \in \text{stab}(E)$ such that:

1. $S_{k+1}^\pi \pi'$, and
2. for every $y \in V_j$, if $\pi \cdot y = y$ then $\pi' \cdot y = y$, and
3. for every $X \subseteq V_{j-1}$ with $S_k \varepsilon X$, $\pi \cdot X = X$ if and only if $\pi' \cdot X = X$.

Proof. Before giving details we suggest why this result is plausible: by our first squashing result Theorem 5.20, we know that we can squash every individual element in nontriv$(\pi)$. We now consider the orbits of $\pi$ and case-split on whether an orbit is finite or infinite:

For a finite orbit we consider it as a finite tuple and squash it into $S_{k+1}$ (meaning $S_{k+1} \varepsilon$ our squashed tuple) using Theorem 5.20.

For an infinite orbit we pick any starting point and squash its elements in succession in a $\mathbb{Z}$-chain (a pair of $\omega$-chains joined at the starting point). Overall this chain is supported by $\bigcup_{i < \omega} S_i$. Now we can squash this chain until it is supported by $S_{k+1}$ — see $\tau$ below. We needed equation (1) in Definition 5.3 precisely to guarantee here a large supply of elements above the chain so that we can build the bijection $\tau$ below.

Now for the details. We need to make some choices and set up some notation:

Let $P$ be the least partition on $V_j$ such that $(y, \pi(y)) \in P$ for every $y \in V_j$ (so $P$ is the orbits of $\pi \in \text{Perm}(V_j)$).

From each $p \in P$ make a fixed but arbitrary choice of representative $y_p \in p$.

If $p \in P$ is infinite then for every $y \in p$ write $\delta(y)$ for the unique $n \geq 0$ such that $y = \pi^n \cdot y_p$ or $y = \pi^{-n} \cdot y_p$; call this the distance of $y$ from $y_p$, so that $\delta(y_p) = 0$.

Consider $p \in P$. It is convenient to consider two cases:

Suppose $p$ is infinite. Then to every $y \in p$ we assign a distinct element $f(y) \in V_{j, \geq 2}$ such that:

1. $S_{k+1}^\delta(y)p \varepsilon f(y)$.
2. $f(y) = \tau_y \cdot y$ for some $\tau_y \in \text{fix}(S_{k+1})$.
3. $(f(y), f(y')) \in E$ for every $y' \in p$.

Suppose $p$ is finite. Then to every $y \in p$ we assign a distinct element $f(y) \in V_{j, \geq 2}$ such that:

1. $S_{k+1}^\delta(y)p \varepsilon f(y)$.
2. $f(y) = \tau_y \cdot y$ for some $\tau_y \in \text{fix}(S_k)$.
3. $(f(y), f(y')) \in E$ for every $y' \in p$.
4. $\tau_y = \tau_{y'}$ for every one of the finitely many $y' \in p$.

By Lemma 5.23 we can do this.

We now derive from $f$ a permutation $\pi'' \in \text{Perm}(V_j)$ as follows:

1. On image$(f) = \{f(y) \mid y \in V_j\}$ the image of $f$.
   We map $f(y) = \tau_y \cdot y \in V_j$ to $f(\pi(y)) = \tau_{\pi(y)} \cdot y$.
2. On $V_j \setminus \text{image}(f)$.
   We map all other $y \in V_j$ to $y$ (so $\pi''$ is the identity here).

Note that by construction $(\bigcup_{i < \omega} S_i)\varepsilon \pi''$ and $\pi'' \in \text{stab}(E)$.

We now choose any $\tau \in \text{Perm}(V_2)$ such that $\tau \in \text{fix}(S_k)$ and $\tau \cdot \bigcup_{i < \omega} S_i = S_{k+1}$, and we set $\pi' = \tau \circ \pi'' \circ \tau$.

It follows using Lemma 3.25 that $S_{k+1}^\pi \cdot \pi'$.

We now prove that $\pi' \cdot X = X$ if and only if $\pi' \cdot X = X$.  

Suppose \( \pi \cdot X = X \) so that also \( \pi^{-1} \cdot X = X \), and consider \( y \in X \) so that also \( \pi \cdot y \in X \). By construction \( \tau \cdot \tau y \cdot y = \pi' \cdot y \). By Lemma 3.27(2) (since \( \tau y, \tau \in \text{fix}(S_k) \)) and \( S_k \cdot X \) \( \pi' \cdot y \in X \).

Conversely if \( \pi' \cdot y \in X \) then we conclude that \( \pi \cdot y \in X \) by applying \( \tau_y^{-1} \circ \tau^{-1} \).

**Proposition 5.25** (The second squashing property: level 1). Suppose \( i \geq 2 \) and \( 1 \leq j \leq i-1 \) and \( E \in EQ_{i, \varnothing} \), and \( k < \omega \) and \( T_k \cdot E \). Suppose \( \pi \in \text{stab}(E) \), so that \( \pi \in \text{Perm}(V_j) \).

Then there exists \( \pi' \in \text{stab}(E) \) such that:

1. \( T_{k+1} \cdot \pi' \), and
2. for every \( y \in V_j \), if \( \pi \cdot y = y \) then \( \pi' \cdot y = y \), and
3. for every \( X \subseteq V_{i-1} \) with \( T_k \cdot X \), \( \pi \cdot X = X \) if and only if \( \pi' \cdot X = X \).

**Proof.** By the same reasoning we used for Proposition 5.24, but working with \( T_k \) instead of \( S_k \).

**5.3.3. Squashing nominal restriction.** We conclude with Proposition 5.28.

**Lemma 5.26.** Suppose \( k < k' < k'' < \omega \). Then:

1. Suppose that:
   
   \(-i \geq 2 \) and \( X \subseteq V_i \).
   
   For every \( \pi \in \text{Perm}(V_2) \) if \( \text{nontriv}(\pi) \subseteq S_{k''} \setminus S_k \) then \( \pi \cdot X = X \).
   
   \( \pi \in \text{Perm}(V_2) \) and \( \text{nontriv}(\pi) \subseteq S_{k''} \setminus S_k \) (so that \( \pi \cdot X = X \)).

   Then
   
   \[ \rho(\pi \cdot S_{k'}) X = \rho S_{k'} X. \]

2. Suppose that:

   \(-i \geq 1 \) and \( X \subseteq V_i \).

   For every \( \pi \in \text{Perm}(V_1) \) if \( \pi \in \text{nontriv}(\pi) \subseteq T_{k''} \setminus T_k \) then \( \pi \cdot X = X \).

   \( \pi \in \text{Perm}(V_1) \) and \( \text{nontriv}(\pi) \subseteq T_{k''} \setminus T_k \) (so that \( \pi \cdot X = X \)).

   Then

   \[ \rho(\pi \cdot T_{k'}) X = \rho T_{k'} X. \]

**Proof.** Either from Definition 3.29 by routine calculations on groups, or as a special case of Lemma 3.33.

**Remark 5.27.** Before proving Proposition 5.28, we give some intuition why it is interesting and important.

In Figures 6 and 7 we will see uses of \( \rho T_k \) and \( \rho S_k \) for some \( k < \omega \), chosen in those figures to be the least ordinal with a certain nice property (\( S_k \cdot X \) for \( X \in V_i, \varnothing_2 \) and \( T_k \cdot X \) for \( X \in V_i, \varnothing_1 \) respectively).

In the context of those figures, Proposition 5.28 suggests that any other choices of \( k < k' < \omega \) would do as well; we could take any \( k' \) such that \( S_{k'} \cdot X \) in Figure 6 (not only the least one), and any \( k' \) such that \( T_{k'} \cdot X \) in Figure 7 (not only the least one)—and this would define the same result. This needs to be proved, of course, and doing so is not trivial. The technical details are given in Lemmas 6.8(2) and 6.16(2).

These Lemmas will be used repeatedly to find a single choice of \( k \) that is suitable for several elements simultaneously. So Proposition 5.28 can be read as the technical result that enables a form of \( \alpha \)-conversion, allowing us where appropriate to ‘freshen’ \( k \) to a ‘fresher’ \( k' \) or \( k'' \) respectively.

**Proposition 5.28(1)** Suppose \( k < k' < \omega \) and \( i \geq 2 \) and \( X \subseteq V_i \) is such that for every \( \pi \in \text{Perm}(V_2) \) if \( \text{nontriv}(\pi) \subseteq S_{k'+1} \setminus S_k \) then \( \pi \cdot X = X \). Then

\[ \rho S_k X = \rho S_{k'} X. \]
Suppose $k < k' < \omega$ and $i \geq 1$ and $X \subseteq V_i$ is such that for every $\pi \in \text{Perm}(V_1)$ if nontriv($\pi$) $\subseteq T_{k'+1}\setminus S_k$ then $\pi \cdot X = X$. Then $\rho T_{k'} \cdot X = \rho T_{k'}.X$.

**Proof.** (1) By construction in Definition 5.3 $T_k \subseteq T_{k'}$ and it follows from Definition 3.29 that $\rho S_{k'}.X \subseteq \rho S_k.X$. So it would suffice to prove that if $\pi \in \text{fix}(S_k)$ then $\pi \cdot \rho S_{k'}.X = \rho S_{k'}.X$.

So suppose $\pi \in \text{fix}(S_k)$. Using Proposition 5.24 for $\pi$ and $k'$, and taking $E$ in that Proposition to be the total equivalence relation $V_2 \times V_2$ so that $S_{k'}.S$ and $\pi \in \text{stab}(E)$ we may assume without loss of generality that nontriv($\pi$) $\subseteq S_{k'+1}\setminus S_k$ (the heavy lifting in the proof is this step).

It follows from Lemma 3.30(1) and our assumption that $\pi \cdot X = X$ that
\[
\pi \cdot \rho S_{k'}.X = \rho(\pi \cdot S_{k'}).X = \rho(\pi \cdot S_{k'}).X.
\]

We now use Lemma 5.26.

(2) Much as for part 1 of this result we can check that it suffices to prove that if $\pi \in \text{fix}(T_k)$ then $\pi \cdot \rho T_{k'}.X = \rho T_{k'}.X$. We use Proposition 5.25 (taking $E$ in that Proposition to be the total equivalence relation $V_1 \times V_1$), Lemma 3.30(1), and Lemma 5.26. \[\square\]

6. BIJECTING LEVELS 1 AND 2

**Remark 6.1.** We bijected $V_1$ and $V_2$ in Theorem 5.1, but as noted in Remark 5.2 this is not enough because the bijection does not extend pointwise to higher levels—the bijection is not symmetric enough and moves us outside the hierarchy of very simple sets.

Now we will exhibit the bijection that does work: it is defined in Figures 6 and 7 and proved bijective in Theorem 6.35.

Diagonalisation problems are avoided because the bijection is partial: we only biject $V_{i+1,32}$ with $V_{i,32}$ (Definition 5.11) and not $V_{i}$ with $V_{i+1}$. However this is enough, because the squashing properties which we proved in Subsection 5.3 guarantee that first-order logic does not notice this partiality (Proposition 7.5).

If the reader found Remark 1.4 helpful, then we can continue the language of that Remark and say that the entropy that we discharge in Theorem 6.35 is $\mathbb{S} \setminus S$, where $S = \{\{x\} \mid x \in V_1\}$.

The proofs in this section are detailed so we take a moment discuss why the overall result (Theorem 6.35) is plausible.

As we noted in Lemma 5.6, from the definitions in Subsection 5.2.1 it follows that
\[
\#(V_1 \setminus \bigcup_{k<\omega} T_k) = \Sigma(\omega+1) = \#(V_2 \setminus \bigcup_{k<\omega} S_k).
\]

Intuitively, an element $x \in V_{i,31}$ is symmetric up to permuting ‘most’ elements in $V_1$, and an element $X \in V_{i+1,32}$ is symmetric up to permuting ‘most’ elements in $V_2$.

By design there is an obvious correspondence $z \mapsto \{z\}$ between elements $z \in T_k$ and elements $\{z\} \in S_k$. It is possible to leverage this correspondence using nominal restriction to move up the hierarchy of very simple sets (details follow in this Section, and see Figures 6 and 7).

As we ascend the hierarchy, extra asymmetric (not-supported-by-any-$S_k$) elements come into existence—as they must, because diagonalisation will be trying break our proof. However, thanks to equation (2) and the squashing results in Subsection 5.3, these extra elements are guaranteed to share a permutation orbit with some symmetric (supported-by-some-$S_k$) element. This is enough to model the first-order quantifiers in TST+; details are in the proofs in Subsection 7.1, most notably for Proposition 7.5.

---

8The parameter $E$ is superfluous here, but it will be important later on in Lemmas 6.30 and 6.31.

9This is a classic nominal small support property, but note that it is *permissive* nominal in the sense of [DGM10], so that the notion of ‘most’ is not directly linked to cardinality. More on this in Remark 5.7.
6.1. Shifting between level 1 and level 2 permutations

We gather together Definitions 6.2 and 6.3. These will be useful in Lemmas 6.6 and 6.14 and the results that depend on them. We will also generalise the Definitions later on once we have more machinery: see Definitions 6.25 and 6.26.

**Definition 6.2.** Suppose $\pi \in \text{Perm}(V_2)$ and suppose nontriv$(\pi) \subseteq S$ (which unpacking Definition 5.3 means that $\pi$ maps singletons to singletons, and fixes non-singleton elements of $V_2$).

Then define $\pi\downarrow \in \text{Perm}(V_1)$ by:

$$\pi\downarrow(x) = x' \quad \text{where} \quad \pi(\{x\}) = \{x'\} \quad \text{for some} \quad x' \in V_1.$$  

**Definition 6.3.** Suppose $\pi \in \text{Perm}(V_1)$. Then define $\pi\uparrow \in \text{Perm}(V_2)$ by:

$$\pi\uparrow(\{x'\}) = \{\pi(x')\} \quad x' \in V_1$$

$$\pi\uparrow(X) = X \quad X \notin S$$

($S = \{\{x\} \mid x \in V_1\} \subseteq V_2$ is from Definition 5.3.)

6.2. Mapping elements up and down

**6.2.1. Mapping down, and injectivity.** Recall the notion $\rho$ of nominal restriction from Definition 3.29:

**Definition 6.4.** Suppose $i \geq 2$ and $X \subseteq V_{i-1}$ and $E \subseteq V_i^2$ (Definition 5.11) and $S_k\cdot S X$ and $S_k\cdot S E$ for some $k' < \omega$. Then define $X\downarrow$ and $E\downarrow$ as in Figure 6.

**Remark 6.5.** This discussion is informal but may be helpful: think of the nominal restriction $\rho$ as analogous to a topological compactification or completion. Then the definition of $X\downarrow$ occurs in two phases:

1. We take the space $X$, call some of its points ‘good’—namely those $x \in X$ such that $S\cdot S X$—and then we form a new space $\{x\downarrow \mid x \in X, S\cdot S x\}$.
   
   (The operation $x \mapsto x\downarrow$ is injective on ‘good’ points, as we shall see later.)

2. Next, we compactify the space by closing it under the action of all permutations $\pi$, including those such that $-(T\cdot \pi\cdot x\downarrow)$.

Viewed in this way, the family of squashing properties in Subsection 5.3, such as Theorem 5.20, can be viewed as saying that ‘good’ points are dense within the larger space.

**Lemma 6.6.** Suppose $\pi \in \text{Perm}(V_1)$ and $T\cdot \pi \cdot \pi$. Suppose and $i \geq 2$ and $X \subseteq V_{i-1}$ and $S_k\cdot S X$ for some $k' < \omega$. Then

$$\pi\cdot (X\downarrow) = (\pi\uparrow \cdot X)\downarrow.$$  

**Proof.** We work by induction on $i$:

— Suppose $i = 2$. By Lemma 5.14 $X = \{x\}$ for some $x \in T_k$ where $k < \omega$. Unpacking Figure 6, the pointwise action (Definition 3.19), and Definition 6.2 we calculate that

$$\pi\cdot \{x\} \downarrow = \pi\cdot \{x\} \downarrow = \pi\cdot x = \pi\cdot \{x\} \downarrow.$$  

---

We do not just write ‘$X \in V_{i,\exists}$’ because we do not wish to assume the existence of $E_0, E_1, \ldots, E_{i-1} \cdot S X$. We describe why in Remark 6.11.
Suppose $i \geq 3$. Let $k < \omega$ be such that $S_k \uparrow X$ and $S_k \uparrow \cdot X$, and $T_k \uparrow \pi$; note that it follows from this and Definition 6.2 that $S_k \uparrow \pi$.

Using the inductive hypothesis and Lemma 5.15, we can apply Lemma 3.33 to derive that

$$(\pi \uparrow \cdot X) \downarrow = \rho T_k \cdot \{ x \downarrow \mid x \in \pi \uparrow \cdot X, T_2 \uparrow x \}$$
and

$$\pi \cdot (X) \downarrow = \pi \cdot \rho T_k \cdot \{ x \downarrow \mid x \in X, T_2 \uparrow x \}.$$

We push $\pi$ inside the restriction in the second expression using Lemma 3.31 (since $T_k \uparrow \pi$) and using the inductive hypothesis (on the $\pi \cdot x \downarrow$) and then Lemma 5.15, we deduce an equality as required. 

**Lemma 6.7.** Suppose $i \geq 2$.

1. If $X \subseteq V_{i-1}$ and $k < \omega$ and $S_k \uparrow X$ and $\pi \in \text{fix}(T_k)$ then $\pi \cdot (X) \downarrow = X \downarrow$.
2. If $E \subseteq V_{i+1}$ and $k < \omega$ and $S_k \uparrow E$ and $\pi \in \text{fix}(T_k)$ then $\pi \cdot (E) \downarrow = E \downarrow$.
3. As a corollary, if $S_k \uparrow X$ then $T_k \uparrow X \downarrow$, and if $S_k \uparrow E$ then $T_k \uparrow E \downarrow$.

**Proof.** We consider just the case of $X$; the case of $E$ is slightly longer but no harder:

1. **Suppose $i = 2$.** Then by Lemma 5.14 $X = \{ x \}$ for some $x \in T_k$, and by Definition 6.4 $X \downarrow = x$.

   We assumed $\pi \in \text{fix}(T_k)$ so $\pi \cdot x = \pi(x) = x$.

2. **Suppose $i \geq 3$.** From Lemma 3.30(2) and the use of $\rho T_k$ in Figure 6.

**Lemma 6.8.** Suppose $i \geq 2$ and $X \subseteq V_{i-1}$ and $E \subseteq V_{i-1} \times V_{i-1}$ and $k < \omega$ and $S_k \uparrow X, E$. Then:

1. $T_k \uparrow \uparrow (X) \downarrow$ for every $k' \geq k$.
2. If $i \geq 2$ then $X \downarrow = \rho T_k \cdot \{ x \downarrow \mid x \in X, S_3 \uparrow x \}$ for any $k' \geq k$.
3. If $i \geq 2$ then $E \downarrow = \rho T_k \cdot \{ (x, x') \mid (x, x') \in E, S_3 \uparrow x, x' \}$ for any $k' \geq k$.

**Proof.** (1) From Lemma 6.7(1) we immediately have that $T_k \uparrow \uparrow (X) \downarrow$. The result follows for all $k' \geq k$ since $T_k \subseteq T_k$.

2. We intend to apply Proposition 5.28, so it would suffice to prove that if $\pi \in \text{Perm}(V_1)$ and $\text{nontriv}(\pi) \subseteq T_{k+1} \setminus T_k$ then

   $$\pi \cdot \{ x \downarrow \mid x \in X, S_3 \uparrow x \} = \{ x \downarrow \mid x \in X, S_3 \uparrow x \}.$$ 

   So choose one such $\pi$. It is a fact of Definition 6.3 that $\text{nontriv}(\pi \uparrow) \subseteq S_{k+1} \setminus S_k$, so $\pi \uparrow \in \text{fix}(S_k)$ and $\pi \uparrow \cdot X = X$. We reason as follows:

   $$\pi \cdot \{ x \downarrow \mid x \in X, S_3 \uparrow x \} = \{ \pi \cdot (x) \downarrow \mid x \in X, S_3 \uparrow x \}$$

   Pointwise action

   $$= \{ (\pi \cdot x) \downarrow \mid x \in X, S_3 \uparrow x \}$$

   $\pi \downarrow \cdot (x \downarrow)$

   $$= \{ x \downarrow \mid x \in \pi \cdot X, S_3 \uparrow x \}$$

   Lemma 5.15

   $$= \{ x \downarrow \mid x \in X, S_3 \uparrow x \}$$

   $X = \pi \uparrow \cdot X$.

3. So we are done.

**Remark 6.9.** We need $\text{nontriv}(\pi) \subseteq T_{k+1} \setminus T_k$ to apply Lemmas 6.6 and 5.15 in the proof of Lemma 6.8(2) above. If we just assumed $\pi \in \text{fix}(T_k)$ then the proof would fail.

Note what the proof is not: let $k < \omega$ be least such that $T_k \uparrow X$, so that $X \downarrow = \rho T_k \cdot \{ x \downarrow \mid x \in X, S_3 \uparrow x \}$. We do not prove Lemma 6.8(2) by choosing some bijection of $V_1$ that bijects $T_k$ with $T_{k'}$ and $V_1 \setminus T_k$ with $V_1 \setminus T_{k'}$, and applying $\tau$ to $X \downarrow$ pointwise. The bijection $\tau$ exists, by Lemma 5.6, but it is not necessarily the case that $\tau \uparrow X = X$.

The actual argument used in Lemma 6.8(2) is far more subtle, and depends on Proposition 5.24 via Proposition 5.28. A similar observation holds of Lemma 6.16.

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11 We would need a version of Proposition 5.28 set up for sets of pairs. This works and would be symbol-for-symbol identical to Proposition 5.28, except for writing $(x, x')$ instead of $x$ as appropriate.

12 ...though we do have a technical result in this spirit: see Lemma 3.33.
If \( X \uparrow = \{ x \} \) \( x \in V_i \) then \( X \uparrow = \rho S_k \cdot \{ x \uparrow \mid x \in X, T_\exists \$ x \} \) \( i > 1, X \in V_i, k < \omega \) least s.t. \( T_k \$ X \).

\( E \uparrow = \rho S_k \cdot \{ (x \uparrow, x' \uparrow) \mid (x, x') \in E, T_\exists \$ x, x' \} \) \( i > 1, E \in \operatorname{EQ}, k < \omega \) least s.t. \( T_k \$ E \).

\[ \text{Fig. 7: Mapping up a level} \quad \text{(Definition 6.12)} \]

\[ \text{Lemma 6.10. Suppose } i \geq 2. \text{ Then:} \]

1. If \( x \in V_{i,32} \) and \( X \subseteq V_i \) and \( S_k \$ X \) for some \( k' < \omega \) then
   \[ x \in X \quad \text{if and only if} \quad x \downarrow \in X \downarrow. \]

2. \( \downarrow \) is injective on \( V_{i,32} \subseteq V_i \).

3. \( \downarrow \) is injective on \( \operatorname{EQ}_{i,32} \subseteq V_i \).

Proof. We reason by induction on \( i \):

1. We consider two implications:
   - If \( x \in X \) then \( x \downarrow \in X \downarrow \) by construction in Definition 6.4.
   - Now suppose \( x \notin X \). Choose \( k < \omega \) such that \( S_k \$ x \), \( X \). Using Lemma 6.8(2)
     \[ X \downarrow = \rho T_k \cdot \{ x \downarrow \mid x \in X, S_\exists \$ x \}. \]
     By Lemma 6.8(1) \( T_k \$ x \downarrow \) and so from Lemma 3.32 \( x \downarrow \in X \downarrow \) if and only if \( x \downarrow \in \{ x \downarrow \mid x \in X, S_\exists \$ x \} \). It follows using part 2 of the inductive hypothesis (for \( i-1 \)) that \( x \in X \).

2. If \( i = 2 \) then we use Lemma 5.14 and the result follows.
   So suppose \( i \geq 3 \) and \( X, X' \in V_{i,32} \) and \( X \neq X' \). Let \( x \in V_{i-1} \) be such that \( x \in X \) and \( x \notin X' \) (the case that \( x \notin X \) and \( x \in X' \) is no harder). There are now two subcases:
   - Suppose \( S_\exists \$ x \). Then using part 1 of this result \( x \downarrow \in X \downarrow \) and \( x \downarrow \notin X' \downarrow \) and so \( X \downarrow \neq X' \downarrow \) as required.
   - Suppose \( \neg (S_\exists \$ x) \). Suppose \( k < \omega \) and \( S_k \$ X, X' \). By Theorem 5.20 there exists \( \pi \in \operatorname{fix}(S_k) \) such that \( S_{k+1} \$ \pi \cdot x \). By Lemma 5.17 \( \pi \cdot x \in X \) and \( \pi \cdot x \notin X' \). Using part 1 of this result we conclude \( X \downarrow \neq X' \downarrow \).

3. Much as for part 2 of this result.

\[ \text{Remark 6.11. It will turn out that } \downarrow \text{ and } \uparrow \text{ from Definitions 6.4 and 6.12 restrict to bijections between } V_{i-1,31} \text{ and } V_{i,32} \text{ for } i \geq 2; \text{ see Theorem 6.35. We define them more generally in Definitions 6.4 and 6.12 because it buys us something useful: the convenience and simplicity of not needing to be within the mutual induction that we start in Definition 6.24, because we do not need to carry an induction with us that e.g. } x \downarrow \in V_{i-2}. \]

6.2.2. Mapping up, and injectivity.

**Definition 6.12.** Suppose \( i \geq 1 \) and \( X \subseteq V_{i-1} \) and \( E \subseteq V_{i-1}^2 \) and \( k < \omega \) and \( T_k \$ X \) and \( T_k \$ E \).
Then define \( X \uparrow \) and \( E \uparrow \) as in Figure 7.

**Remark 6.13.** We continue the informal intuition from Remark 6.5. The definition of \( X \uparrow \) is visibly analogous to that of \( X \downarrow \), with one slight extra subtlety: when we complete with \( \rho S_k \) we are closing under \( \pi \) that may map singleton elements of \( V_2 \) to non-singleton elements in \( V_2 \). Intuitively this will makes no practical difference because of (2) in Remark 6.1 (which follows from Lemma 5.6(3&5)).

**Lemma 6.14.** Suppose \( \pi \in \operatorname{Perm}(V_2) \) and \( S_\exists \$ \pi \). Suppose and \( i \geq 1 \) and \( X \subseteq V_{i-1} \) and \( T_k \$ X \) for some \( k' < \omega \). Then

\[ \pi \cdot (X \uparrow) = (\pi \downarrow \cdot X) \uparrow. \]

Proof. We work by induction on \( i \):
Suppose \( i = 1 \). Then unpacking Figure 7 and the pointwise action (Definition 3.19)
\[
(\pi \downarrow \cdot X) \uparrow = \{ \pi \downarrow (X) \} = \pi \cdot \{ X \} = \pi \cdot X \uparrow
\]
so we are done.

Suppose \( i \geq 2 \). Let \( k < \omega \) be such that \( T_k \cdot X \) and \( T_k \cdot X \uparrow \), and \( S_k \cdot \pi \); note that it follows from this and Definition 6.3 that \( T_k \cdot \pi \).

Using the inductive hypothesis and Lemma 5.15, we can apply Lemma 3.33 to derive that
\[
(\pi \downarrow \cdot X) \uparrow = \rho S_k \cdot \{ x \uparrow \mid x \in \pi \downarrow \cdot X, \; T_\exists \cdot x \} \quad \text{and} \quad \pi \cdot (X \uparrow) = \pi \cdot \rho S_k \cdot \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \}. \]

We push \( \pi \) inside the restriction in the second expression using Lemma 3.31 (since \( S_k \cdot \pi \)) and using the inductive hypothesis (on the \( \pi \cdot x \uparrow \)) and then Lemma 5.15, we deduce an equality as required.

**Lemma 6.15.** Suppose \( i \geq 1 \).

(1) If \( X \subseteq V_{i-1} \) and \( k < \omega \) and \( T_k \cdot X \) and \( \pi \in \text{fix}(S_k) \) then \( \pi \cdot (X \uparrow) = X \uparrow \).

(2) If \( E \subseteq V_{i-1} \) and \( k < \omega \) and \( T_k \cdot E \) and \( \pi \in \text{fix}(S_k) \) then \( \pi \cdot (E \uparrow) = E \uparrow \).

(3) As a corollary, if \( T_k \cdot X \) then \( S_k \cdot X \uparrow \), and if \( T_k \cdot E \) then \( S_k \cdot E \uparrow \).

**Proof.** We consider just the case of \( X \); the case of \( E \) is slightly longer but no harder:

(1) Suppose \( i = 1 \). Then by Definition 6.12 \( X \uparrow = \{ X \} \). We assumed \( \pi \in \text{fix}(T_k) \) so \( \pi \cdot \{ X \} = \{ \pi (X) \} = \{ X \} \).

(2) Suppose \( i \geq 2 \). From Lemma 3.30(2) and the use of \( \rho S_k \) in Figure 7.

**Lemma 6.16.** Suppose \( i \geq 1 \) and \( X \subseteq V_{i-1} \) and \( E \subseteq V_{i-1} \times V_{i-1} \) and \( k < \omega \) and \( T_k \cdot X, \; E \). Then:

(1) \( S_k \cdot \{ X \uparrow \} \) for every \( k' \geq k \).

(2) If \( i \geq 2 \) then \( X \uparrow = \rho S_{k'} \cdot \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \} \) for any \( k' \geq k \).

(3) If \( i \geq 2 \) then \( E \uparrow = \rho S_{k'} \cdot \{ (x \uparrow, x' \uparrow) \mid (x, x') \in E, \; T_\exists \cdot x, \; x' \} \) for any \( k' \geq k \).

**Proof.** (1) From Lemma 6.15(1) we immediately have that \( S_k \cdot \{ X \uparrow \} \). The result follows for all \( k' \geq k \) since \( S_k \subseteq S_{k'} \).

(2) We intend to apply Proposition 5.28, so it would suffice to prove that if \( \pi \in \text{Perm}(V_2) \) and \( \text{nontriv}(\pi) \subseteq S_{k'+1} \setminus S_k \) then
\[
\pi \cdot \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \} = \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \}.
\]

So choose one such \( \pi \). It is a fact of Definition 6.2 that \( \text{nontriv}(\pi \uparrow) \subseteq S_{k'+1} \setminus S_k \), so \( \pi \downarrow \in \text{fix}(T_k) \) and \( \pi \downarrow \cdot X = X \). We reason as follows:
\[
\pi \cdot \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \} = \{ \pi \cdot (x \uparrow) \mid x \in X, \; T_\exists \cdot x \} \quad \text{Pointwise action}
= \{ (\pi \downarrow \cdot x) \uparrow \mid x \in X, \; T_\exists \cdot x \} \quad \text{Lemma 6.14}
= \{ x \uparrow \mid x \in \pi \downarrow \cdot X, \; T_\exists \cdot x \} \quad \text{Lemma 5.15}
= \{ x \uparrow \mid x \in X, \; T_\exists \cdot x \} \quad X = \pi \downarrow \cdot X.
\]

So we are done.

(3) The argument for \( E \) is just as for \( X \).\(^{13}\)

**Lemma 6.17.** Suppose \( i \geq 2 \). Then:

(1) If \( x \in V_{i-1, 31} \) and \( X \subseteq V_i \) and \( T_{k'} \cdot X \) for some \( k' < \omega \) then
\[
x \in X \quad \text{if and only if} \quad x \uparrow \in X \uparrow.
\]

(2) \( \uparrow \) is injective on \( V_{i-1, 31} \subseteq V_{i-1} \).

\(^{13}\)We would need a version of Proposition 5.28 set up for sets of pairs. This works and would be symbol-for-symbol identical to Proposition 5.28, except for writing \( (x, x') \) instead of \( x \) as appropriate.
(3) $\uparrow$ is injective on $\text{EQ}_{i-1, \exists 1} \subseteq \text{EQ}_{i-1}$.

**Proof.** We reason by induction on $i$:

(1) We consider two implications:
- If $x \in X$ then $x\uparrow \in X\uparrow$ by construction in Definition 6.12.
- If $x \not\in X$. Choose $k < \omega$ such that $T_k \cdot x \in X$. Using Lemma 6.16(2)
  \[ X\uparrow = \rho S_k \cdot \{x\uparrow \mid x \in X, T_\exists \cdot x\} \]

  By Lemma 6.16(1) $S_k \cdot x\uparrow$ and so from Lemma 3.32 $x\uparrow \in X\uparrow$ if and only if $x\uparrow \in \{x\uparrow \mid x \in X, T_\exists \cdot x\}$.

  It follows using part 2 of the inductive hypothesis (for $i-1$) that $x \in X$.

(2) If $i = 2$ then we note that $\{x\} = \{x'\}$ if and only if $x = x'$, and the result follows.

  So suppose $i \geq 3$ and $X, X' \in V_{i, \exists 1}$ and $X \neq X'$. Let $x \in V_{i-1}$ be such that $x \in X$ and $x \not\in X'$.

  (the case that $x \not\in X$ and $x \in X'$ is no harder). There are now two subcases:

- **Suppose $T_\exists \cdot x$.** Then using part 1 of this result $x\uparrow \in X\uparrow$ and $x\uparrow \not\in X'\uparrow$ and so $X\uparrow \neq X'\uparrow$ as required.

- **Suppose $(T_\exists \cdot x)$.** Suppose $k < \omega$ and $T_k \cdot X, X'$. By Theorem 5.20 there exists $\pi \in \text{fix}(T_k)$ such that $T_{k+1} \cdot \pi \cdot x$. By Lemma 5.17 $\pi \cdot x \in X$ and $\pi \cdot x \not\in X'$. Using part 1 of this result we conclude that $X\uparrow \neq X'\uparrow$.

(3) Much as for part 2 of this result. □

**6.2.3. Permutations at levels 0 and 1.** Recall from Definition 5.3 that $S_k = \{x\mid x \in T_k\}$ for every $k < \omega$. Lemmas 6.18 and 6.19 are in a similar spirit to Lemma 3.31:

**Lemma 6.18.** Suppose $\pi \in \text{Perm}(V_0)$ and $k < \omega$. Then:

1. If $i \geq 1$ and $X \subseteq V_i$ then $\pi \cdot (\rho T_k \cdot X) = \rho T_k \cdot \pi \cdot X$.
2. If $i \geq 2$ and $X \subseteq V_i$ then $\pi \cdot (\rho S_k \cdot X) = \rho S_k \cdot \pi \cdot X$.

**Proof.** By routine calculations using equivariance of $T_k$ and consequently of $S_k$ over $V_0$ (condition 3 of Definition 5.3).

**Lemma 6.19.** Suppose $\pi \in \text{Perm}(V_1)$ and $i \geq 2$ and $X \subseteq V_i$ and $k < \omega$ and $T_k \cdot \pi$ (so that nontriv($\pi$) $\subseteq T_k$). Then

\[ \pi \cdot (\rho S_k \cdot X) = \rho S_k \cdot \pi \cdot X. \]

**Proof.** By calculations on groups, noting from Definition 5.3 that $S_k = \{x\mid x \in T_k\}$.

**Lemma 6.20.** Suppose $\pi \in \text{Perm}(V_0)$ and $k < \omega$. Then:

1. If $i \geq 1$ and $X \subseteq V_i$ then $T_k \cdot \pi \cdot X$ if and only if $T_k \cdot \pi \cdot X$.
2. If $i \geq 2$ and $X \subseteq V_i$ then $S_k \cdot \pi \cdot X$ if and only if $S_k \cdot \pi \cdot X$.

**Proof.** A consequence of equivariance of $T_k$ and consequently of $S_k$ over $V_0$ (condition 3 of Definition 5.3).

**Lemma 6.21.** Suppose $\pi \in \text{Perm}(V_0)$ and $i \geq 2$ and $X \subseteq V_{i, \exists 2}$ and $E \in \text{EQ}_{i, \exists 2}$. Then:

1. $(\pi \cdot X)\downarrow = \pi \cdot (X\downarrow)$.
2. If $E_0 \in \text{EQ}_0$ then $E_0 \cdot X$ implies $E_0 \cdot X\downarrow$.
3. If $E_0 \in \text{EQ}_0$ then $E_0 \cdot E$ implies $E_0 \cdot E\downarrow$.

**Proof.** (1) By a routine induction on $i$ in Figure 6, using Lemmas 6.18 and 6.20.

(2) Suppose $E_0 \cdot X$, meaning by Definition 3.20(2) that if $\pi \in \text{stab}(E_0)$ then $\pi \cdot X = X$. We use part 2 of this result to deduce that if $\pi \in \text{stab}(E_0)$ then $\pi \cdot (X\downarrow) = X\downarrow$.

(3) The argument is just as for $X$.

---

\[ ^{14} \text{In fact it would suffice that } X \subseteq V_{i-1} \text{ and } S_k \cdot X \text{ for some } k' < \omega, \text{ but we will not need this generality.} \]
Lemma 6.22. Suppose $\pi \in \text{Perm}(V_0)$ and $i \geq 1$ and $X \in V_{i, \exists 1}$ and $E \in \text{EQ}_{i, \exists 1}$. Then:

1. $(\pi \cdot X)^\uparrow = \pi \cdot (X)^\uparrow$.
2. If $E_0 \in \text{EQ}_0$ then $E_0 \cdot X$ implies $E_0 \cdot X^\uparrow$.
3. If $E_0 \in \text{EQ}_0$ then $E_0 \cdot E$ implies $E_0 \cdot E^\uparrow$.

Proof. Just as for Lemma 6.21. \qed

Lemma 6.23 resembles Lemma 6.14 and has a similar proof:

Lemma 6.23. Suppose $\pi \in \text{Perm}(V_1)$ and $T_3 \cdot \pi$. Suppose $i \geq 1$ and $X \in V_{i, \exists 1}$ and $E \in \text{EQ}_{1, \exists 1}$. Then:

1. $(\pi \cdot X)^\uparrow = \pi \cdot (X)^\uparrow$.
2. If $i \geq 2$ and $E_1 \in \text{EQ}_1$ and $E_1 \cdot X$ then $E_1 \cdot X^\uparrow$.
3. If $i \geq 2$ and $E_1 \in \text{EQ}_1$ and $E_1 \cdot E$ then $E_1 \cdot E^\uparrow$.

Proof. Part 2 follows from part 1 by unpacking Definition 3.20 and performing a small calculation using Lemma 6.16(2) (since we assumed $T_3 \cdot \pi$). Part 3 is by further calculations and is no harder.

For part 1 we work by induction on $i$:

— Suppose $i = 1$. Then unpacking Figure 7 and the pointwise action (Definition 3.19)

$$ (\pi \cdot X)^\uparrow = \{\pi(X)\} = \pi \{X\} = \pi \cdot (X)^\uparrow $$

so we are done.

— Suppose $i \geq 2$. By assumption $T_3 \cdot \pi, X$ so by Lemma 5.15 also $T_3 \cdot \pi \cdot X$. Let $k < \omega$ be such that $T_k \cdot X, \pi \cdot X, \pi$. By Lemma 6.16(2) we have

$$ (\pi \cdot X)^\uparrow = \rho S_k \cdot \{x^\uparrow \mid x \in \pi \cdot X, T_3 \cdot x\} \quad \text{and} \quad \pi \cdot (X)^\uparrow = \rho S_k \cdot \{x^\uparrow \mid x \in X, T_3 \cdot x\}.$$

We simplify the second expression using Lemma 6.19 (since $T_k \cdot \pi$):

$$ \pi \cdot (X)^\uparrow = \rho S_k \cdot \{\pi \cdot (x)^\uparrow \mid x \in X, T_3 \cdot x\}.$$

We use the inductive hypothesis on $\pi \cdot (x)^\uparrow$, and then Lemma 5.15, and we deduce an equality as required. \qed

6.3. Bijectivity

Definition 6.24. We conduct all definitions and results between this Definition and Theorem 6.35, inductively on $i \geq 2$. So this subsection is one big mutual induction on $i$, and we can appeal to Theorem 6.35 so long as we only do so for lower values of $i$.

Recall $V_0$ from Definition 4.3 and $V_{i, \exists 1}$ and $V_{i, \exists 2}$ from Definition 5.11. Definition 6.25 extends Definition 6.2 to higher levels:

Definition 6.25. Suppose $i \geq 3$ and $2 \leq j \leq i - 1$ and $\pi \in \text{Perm}(V_j)$ and $S_3 \cdot \pi$ (Definition 5.12). Define $\pi \downarrow \in \text{Perm}(V_{j - 1})$ as follows, for each $x \in V_{j - 1}$:

— If $x \in V_{j - 1, \exists 1}$ then

$$ \pi \downarrow(x) = (\pi \cdot (x)^\uparrow) \downarrow. $$

— If $x \in V_{j - 1} \setminus V_{j - 1, \exists 1}$ then

$$ \pi \downarrow(x) = x. $$

Definition 6.26 extends Definition 6.3 to higher levels:

Definition 6.26. Suppose $i \geq 3$ and $2 \leq j \leq i - 1$ and $\pi \in \text{Perm}(V_{j - 1})$ and suppose $T_3 \cdot \pi$ (Definition 5.12). Define $\pi \uparrow \in \text{Perm}(V_j)$ as follows, for each $x \in V_j$:
— If \( x \in V_{j,32} \) then
\[
\pi^\uparrow(x) = (\pi \cdot (x^\downarrow))^\uparrow.
\]

— If \( x \in V_j \setminus V_{j-1,32} \) then
\[
\pi^\uparrow(x) = x.
\]

Remark 6.27(1) Definitions 6.26 and 6.25 will only be needed once our mutual induction reaches \( i = 3 \). In the proofs that follow—for instance in Lemma 6.30(1)—the case of \( i = 2 \) (where there is one) will not appeal to \( \pi^\uparrow \) or \( \pi^\downarrow \).

(2) For the special case of \( \pi \in \text{Perm}(V_2) \), Definition 6.25 simplifies to Definition 6.2.

(3) For the special case of \( \pi \in \text{Perm}(V_1) \), Definition 6.26 simplifies to Definition 6.3.

Lemma 6.28 is a basic sanity check:

Lemma 6.28. Suppose \( i \geq 3 \) and \( 2 \leq j \leq i-1 \). Then:

1. If \( \pi \in \text{Perm}(V_j) \) and \( x \in V_{j-1} \) then \( \pi^\downarrow(x) \in V_{j-1} \).

As a corollary, \( \pi^\downarrow \) is indeed a bijection (a permutation) on \( \text{Perm}(V_{j-1}) \).

2. If \( \pi \in \text{Perm}(V_{j-1}) \) and \( x \in V_j \) then \( \pi^\uparrow(x) \in V_j \).

As a corollary, \( \pi^\uparrow \) is indeed a bijection (a permutation) on \( \text{Perm}(V_j) \).

Proof. By routine calculations from Theorem 6.35(1) (for levels \( j-1 \) and \( j \)).

Lemma 6.29 extends Lemmas 6.6 and 6.14:

Lemma 6.29. Suppose \( i \geq 3 \) and \( 2 \leq j \leq i-1 \). Then:

1. Suppose \( j-1 \leq i' \leq i-1 \) and \( X \in V_{i',31} \) and \( \pi \in \text{Perm}(V_j) \) and \( S_3 \$ \pi \). Then
\[
(\pi^\downarrow \cdot X)^\uparrow = \pi \cdot (X^\uparrow).
\]

2. Suppose \( j \leq i' \leq i-1 \) and \( X \in V_{i',32} \) and \( \pi \in \text{Perm}(V_{j-1}) \) and \( T_3 \$ \pi \). Then
\[
(\pi^\uparrow \cdot X)^\downarrow = \pi \cdot (X^\downarrow).
\]

Proof. For part 1 we reason by induction on \( i' \) (if \( j = 2 \) then we could also use Lemma 6.14 directly):

— Suppose \( i' = j-1 \). We use Definition 6.25 directly.

— Suppose \( i' \geq j \). Then let \( k < \omega \) be such that \( T_k \$ X \) and \( S_k \$ \pi \cdot X \) and \( S_k \$ \pi \). Using Lemma 5.15(1) and the inductive hypothesis, we see that we can apply Lemma 3.33 to derive that
\[
(\pi^\downarrow \cdot X)^\uparrow = \rho S_k \cdot \{ x^\uparrow \mid x \in \pi^\downarrow \cdot X, T_3 \$ x \} \quad \text{and} \quad \pi \cdot (X^\uparrow) = \pi \cdot \rho S_k \cdot \{ x^\uparrow \mid x \in X, T_3 \$ x \}.
\]

We simplify the second expression using Lemma 3.31 (since \( S_k \$ \pi \)):
\[
\pi \cdot (X^\uparrow) = \rho S_k \cdot \pi \cdot (x^\uparrow) \mid x \in X, T_3 \$ x \}.
\]

Using the inductive hypothesis (on \( \pi \cdot (x^\uparrow) \)) and then Lemma 5.15(1), we deduce an equality as required.

For part 2 we reason much as for part 1, using Definition 6.26 and Lemma 5.15(2).

Lemma 6.30(1) If \( i \geq 2 \) and \( X \in V_{i,32} \) then \( X^\downarrow \in V_{i-1,31} \).

2. If \( i \geq 3 \) and \( 2 \leq j \leq i-1 \) and \( E \in \text{EQ}_{j,32} \) then \( E^\downarrow \in \text{EQ}_{j-1,31} \).

3. If \( i \geq 4 \) and \( X \in V_{i,32} \) and \( 2 \leq j \leq i-2 \) and \( E \in \text{EQ}_{j,32} \) then \( E \$ X \) implies \( E^\downarrow \$ X^\downarrow \).

4. If \( i \geq 3 \) and \( E' \in \text{EQ}_{i,31} \) and \( 2 \leq j \leq i-1 \) and \( E \in \text{EQ}_{j,31} \) then \( E \$ E' \) implies \( E^\downarrow \$ E'^\downarrow \).

Proof. We reason as follows:

1. — Suppose \( i = 2 \) so that \( X \in V_2 \). Using Lemma 5.14 \( X = \{ x \} \) for some \( x \in T_k \), and from Figure 6 \( X^\downarrow = x \), so we are done.
— Suppose $i = 3$ and suppose (since $X$ is very simple; Definition 4.3) that $X$ is supported by $E_0, E_1$. By Lemma 6.21 $E_0 \S X \diamondsuit$.

— Suppose $i \geq 4$ and suppose (since $X$ is very simple; Definition 4.3) that $X$ is supported by $E_0, E_1, \ldots, E_{i-2}$. By Lemma 6.21 $E_0 \S X \downarrow$. By part 2 of this result (for $2, 3, \ldots, i-2$) $E_2 \downarrow \in \mathcal{E}_1, \ldots, E_{i-2} \downarrow \in \mathcal{E}_{i-3}$.

By part 3 of this result these equivalence relations support $X \downarrow$.

(2) By Theorem 6.35(1) the assignment $(x, x') \mapsto (x \downarrow, x' \downarrow)$ is injective on pairs $(x, x') \in V^2_{j,32}$ where $j \leq i-1$; so $\{(x \downarrow, x' \downarrow) \mid (x, x') \in E\}$ is a small equivalence relation on $V_{j-1,31} \equiv \{x \downarrow \mid x \in V_{j,32}\}$. It follows by a short argument using Theorem 5.20 and Lemma 6.8(3) that $E \downarrow$ is a small (Definition 3.14) equivalence relation on $V_{j-1}$.

We also need to verify that $E \downarrow$ is very simple. We argue just as for part 1 above, where in the final step we argue that by part 4 of this result these equivalence relations support $E \downarrow$.

(3) Consider $\pi \in \text{stab}(E \downarrow)$, so that $\pi \in \text{Perm}(V_{j-1})$ where $j \leq i-2$ and $(y, \pi(y)) \in E \uparrow$ for every $y \in V_{j-1}$. We wish to show that $\pi \cdot (X \downarrow) = X \downarrow$.

Choose $k < \omega$ such that $S_k \S X$ (since we assumed $X \in V_{i,32}$) and $T_k \S E \downarrow$ (since we assumed $E \in \mathcal{E}_{j,32}$, using Lemma 6.7). By Proposition 5.25 (since by Theorem 6.35(2) $E \downarrow \in \mathcal{E}_{j-1,31}$) we may assume without loss of generality that $T_{k+1} \S \pi$ (Definition 5.12).

By Lemma 6.29(2) $\pi \cdot (X \downarrow) = (\pi \cdot X \downarrow)$. By calculations using Theorem 6.35 $\pi \cdot X \downarrow \in \text{stab}(E)$ and since we assumed $E \S X$ we have $\pi \cdot X \downarrow = X \downarrow$. Thus $\pi \cdot (X \downarrow) = X \downarrow$ as required.

(4) As for part 3, but for a set of pairs.

Lemma 6.31(1) If $i \geq 2$ and $X \in V_{i-1,31}$ then $X \uparrow \subseteq V_{i,32}$.

(2) If $i \geq 3$ and $2 \leq j \leq i-1$ and $E \in \mathcal{E}_{j-1,31}$ then $E \uparrow \in \mathcal{E}_{j,32}$.

(3) If $i \geq 4$ and $X \in V_{i-1,31}$ and $2 \leq j \leq i-2$ and $E \in \mathcal{E}_{j-1,31}$ then $E \S X$ implies $E \uparrow \S X \uparrow$.

(4) If $i \geq 3$ and $E^* \in \mathcal{E}_{i-1,31}$ and $2 \leq j \leq i-1$ and $E \in \mathcal{E}_{j-1,31}$ then $E \S E^*$ implies $E \uparrow \S E^* \uparrow$.

Proof. We reason as follows:

(1) — Suppose $i = 2$ so $X \in V_1$. From Figure 7 $X \uparrow = \{X\}$, so we are done.

— Suppose $i = 3$ and suppose (since $X \in V_2$ is very simple; Definition 4.3) that $X$ is supported by $E_0$. From Lemma 6.22(2) $E_0 \S X \uparrow$. From Lemma 6.23(2) $E_1 \S X \uparrow$, where we set $E_1 = \{X, V_1 \setminus X\}$.

— Suppose $i \geq 4$ and suppose (since $X \in V_{i-1}$ is very simple) that $X$ is supported by $E_0, E_1, \ldots, E_{i-3}$. From Lemma 6.22(2) $E_0 \S X \uparrow$. From Lemma 6.23(2) $E_1 \S X \uparrow$.

By part 3 of the inductive hypothesis $E_1 \uparrow \in \mathcal{E}_2, E_2 \uparrow \in \mathcal{E}_3, \ldots, E_{i-3} \uparrow \in \mathcal{E}_{i-2}$, and by part 2 of the inductive hypothesis these equivalence relations support $X \uparrow$, so we are done.

(2) By Theorem 6.35(1) the assignment $(x, x') \mapsto (x \uparrow, x' \uparrow)$ is injective on pairs $(x, x') \in V^2_{j-1,31}$ where $j \leq i-1$; so $\{(x \uparrow, x' \uparrow) \mid (x, x') \in E\}$ is a small equivalence relation on $V_{j,31} \equiv \{x \uparrow \mid x \in V_{j,32}\}$. It follows by a short argument using Theorem 5.20 and Lemma 6.16(3) that $E \uparrow$ is a small (Definition 3.14) equivalence relation on $V_{j}$.

We also need to verify that $E \uparrow$ is very simple. We argue just as for part 1 above, where in the final step we argue that by part 4 of this result these equivalence relations support $E \uparrow$.

(3) Consider $\pi \in \text{stab}(E \uparrow)$, so that $\pi \in \text{Perm}(V_{j+1})$ where $j \leq i-2$ and $(y, \pi(y)) \in E \uparrow$ for every $y \in V_{j+1}$. We wish to show that $\pi \cdot (X \uparrow) = X \uparrow$.

Choose $k < \omega$ such that $S_k \S X$ (since we assumed $X \in V_{i,31}$) and $S_k \S E \uparrow$ (since we assumed $E \in \mathcal{E}_{j,31}$, using Lemma 6.15). By Proposition 5.24 (since by Theorem 6.35(2) $E \uparrow \in \mathcal{E}_{j,32}$) we may assume without loss of generality that $S_{k+1} \S \pi$ (Definition 5.12).

By Lemma 6.29(1) $\pi \cdot (X \uparrow) = (\pi \cdot X \uparrow) \uparrow$. By calculations using Theorem 6.35 $\pi \downarrow \in \text{stab}(E)$ and since we assumed $E \S X$ we have $\pi \downarrow \cdot X = X \uparrow$. Thus $\pi \cdot (X \uparrow) = X \uparrow$ as required.

(4) As for part 3, but for a set of pairs.

Proposition 6.32(1) Suppose $i \geq 2$ and $X \in V_{i,32}$. Then $X \downarrow = X$. 

(2) Suppose \( i \geq 3 \) and \( E \in \text{EQ}_{i-1,32} \). Then \( E\downarrow = E \).

**Proof.** For part 1, we reason as follows:

— Suppose \( i = 2 \).
  By Lemma 5.14 \( X = \{x\} \) for some \( x \in T_k \). Unpacking Figures 6 and 7, \( X\downarrow = x = \{x\} = X \).

— Suppose \( i \geq 3 \).
  By Lemmas 6.30(1) and 6.31(1) \( X\downarrow \in \mathcal{V}_{i,32} \). From Corollary 5.21(2) it suffices to prove
  \[ \forall x' \in \mathcal{V}_{i,32} \left( x' \in X \iff x' \in X\downarrow \right). \]
  So choose \( x' \in \mathcal{V}_{i-1,32} \) and suppose \( x' \in X\downarrow \). By inductive hypothesis (for \( i-1 \)) \( x' = (x'\downarrow)\downarrow \). Using Lemmas 6.17(1) and 6.10(1) \( x' \in X \) as required.

The reverse implication is no harder.

Part 2 is just the same, using the cases for \( E \) in the same results as required.

\[ \square \]

**Remark 6.33.** In Proposition 6.32(2) (also in Proposition 6.34) we take \( i \geq 3 \) and then work with \( \text{EQ}_{i-2,32} \) and \( \text{EQ}_{i-1,32} \). Why not just take \( i \geq 2 \) and work with \( \text{EQ}_{i-1,32} \) and \( \text{EQ}_{i,32} \)? This is an artefact of how the levels work in the big mutual induction that we began in Definition 6.24, and indicates that we do not need Proposition 6.32(2) to prove Theorem 6.35 for the case \( i = 2 \).

It might also be helpful to express the idea behind the proof of Proposition 6.32 in an equational style. What follows is not a formal proof but it may throw some light on what the formal proof above is doing:

\[
x \in X\downarrow = \rho S_k \cdot \{x' \mid x' \in X\downarrow, T_3 \uparrow x'\}
= \rho S_k \cdot \{x' \mid x' \in \rho T_k \cdot \{x \mid x \in X, S_3 \uparrow x\}, T_3 \uparrow x'\}
= \rho S_k \cdot \{x \mid x \in X, S_3 \uparrow x\}
= \rho S_k \cdot X
= X
\]

**Proposition 6.34(1)** Suppose \( i \geq 2 \) and \( X \in \mathcal{V}_{i-1,31} \). Then \( X\downarrow = X \).

**Proposition 6.34(2)** Suppose \( i \geq 3 \) and \( E \in \text{EQ}_{i-2,32} \). Then \( E\downarrow = E \).

**Proof.** For part 1 we reason as follows:

— Suppose \( i = 2 \), so \( X \in \mathcal{V}_1 \).
  Unpacking Figures 6 and 7, \( X\downarrow = \{x\}\downarrow = X \).

— Suppose \( i \geq 3 \).
  By Lemmas 6.31(1) and 6.30(1) \( X\downarrow \in \mathcal{V}_{i-1,31} \). From Corollary 5.21(1) it suffices to prove
  \[ \forall x' \in \mathcal{V}_{i-2,31} \left( x' \in X \iff x' \in X\downarrow \right). \]
  So choose \( x' \in \mathcal{V}_{i-2,31} \) and suppose \( x' \in X\downarrow \). By inductive hypothesis (for \( i-1 \)) \( x' = (x'\downarrow)\downarrow \). Using Lemmas 6.10(1) and 6.17(1) \( x' \in X \) as required.

The reverse implication is no harder.

Part 2 is just the same, using the cases for \( E \) in the same results as required.

\[ \square \]

Theorem 6.35 does not biject all of \( \mathcal{V}_1 \) with all of \( \mathcal{V}_2 \) (Definition 4.3); it bijects \( \mathcal{V}_{1,31} \subseteq \mathcal{V}_1 \) with \( \mathcal{V}_{2,32} \subseteq \mathcal{V}_2 \) (Definition 5.11). We discussed why this is interesting in Remark 6.1. We will use Theorem 6.35 just once below, in Corollary 7.7:

**Theorem 6.35(1)** If \( i \geq 2 \) then \( \uparrow \) and \( \downarrow \) form a bijection of \( \mathcal{V}_{i-1,31} \) with \( \mathcal{V}_{i,32} \).

**Theorem 6.35(2)** If \( i \geq 3 \) then \( \uparrow \) and \( \downarrow \) form a bijection of \( \text{EQ}_{i-1,31} \) with \( \text{EQ}_{i,32} \).

**Proof.** (1) We combine Lemmas 6.30(1) and 6.31(1) with Propositions 6.32 and 6.34.

(2) We combine Lemmas 6.30(2) and 6.31(2) with Propositions 6.32 and 6.34.

\[ \square \]
7. TYPICAL AMBIGUITY

7.1. Permutations acting on closed predicates

**Notation 7.1.** Suppose $\phi \in \text{Pred}$ is a predicate and $\pi \in \text{Perm}(V_i)$ is a permutation. Write $\pi \cdot \phi$ for that predicate obtained by replacing each constant $x$ that appears in $\phi$, with $\pi \cdot x$.

**Lemma 7.2.** Suppose $\phi \in \text{CPred}$ is a closed predicate (so $fv(\phi) = \emptyset$, but $\phi$ may mention constants). Then:

1. If $\pi \in \text{Perm}(V_1)$ then $\models \phi$ if and only if $\models \pi \cdot \phi$.
2. If $\text{minlev}(\phi) \geq 2$ (Definition 2.4) and $\pi \in \text{Perm}(V_2)$ then $\models \phi$ if and only if $\models \pi \cdot \phi$.

**Proof.** By routine inductions on $\phi$ using the fact that the permutation $\pi$ acts bijectively.

7.2. A pair of denotations

**Definition 7.3.** Suppose $\phi \in \text{CPred}$ is a closed predicate (so $fv(\phi) = \emptyset$, but $\phi$ may mention constants), and suppose $k < \omega$. Then define $k \models^1 \phi$ and $k \models^2 \phi$ as in Figure 8.

**Notation 7.4.** It will be useful to mildly generalise Definition 3.24. Suppose $X$ is some finite collection of elements and suppose $k < \omega$. Then write:

$\quad$— $T_k \cdot X$ when $T_k \cdot X$ for every $X \in X$.
$\quad$— $S_k \cdot X$ when $S_k \cdot X$ for every $X \in X$.
$\quad$— Similarly for $T_\exists$ and $S_\exists$.

**Proposition 7.5.** Suppose $\phi \in \text{CPred}$ is a closed predicate. Then:

1. If $\text{minlev}(\phi) \geq 1$ and $T_k \cdot \text{consts}(\phi)$ then $\models \phi$ if and only if $k \models^1 \phi$.
2. If $\text{minlev}(\phi) \geq 2$ and $S_k \cdot \text{consts}(\phi)$ then $\models \phi$ if and only if $k \models^2 \phi$.

15This generalisation is identical to what we get if we just give $X$ the pointwise action as a finite set. So we can view this as a generalisation, or as a lemma about finite powersets. The relevant mathematics is presented in some generality in [Gab11, Theorem 2.29].
Proof. We work by induction on \( \phi \). The interesting case for both parts is for \( \exists a.\phi \). We consider part 1 (for \( \vdash 1 \)), part 2 is identical:

— Suppose \( \vdash \exists a.\phi \). So \( \vdash \phi[a:=x] \) for some \( x \in V_{\text{level}(a)} \).

From Theorem 5.20 we obtain a \( \tau \in \text{fix}(T_k) \) such that

— \( T_{k+1} \tau \vdash \tau \cdot x \) (so that \( \tau \cdot x \in V_{\text{level}(a),\exists 1} \)), and

— \( \tau \cdot c = c \) for every constant \( c \in \text{consts}(\phi) \) (because \( T_k \tau \vdash \tau \in \text{fix}(T_k) \)).

Using Lemma 7.2 we have that \( \vdash \phi[a:=\tau \cdot x] \). By inductive hypothesis \( k+1 \vdash 1 \phi[a:=\tau \cdot x] \), and following Figure 8 \( k \vdash 1 \exists a.\phi \) as required.

— Suppose \( \vdash 1 \exists a.\phi \). So \( \vdash \phi[a:=x] \) for some \( x \in V_{\text{level}(a),\exists 1} \).

Using the inductive hypothesis, \( \vdash \phi[a:=x] \) for some \( x \in V_{\text{level}(a)} \). Thus \( \vdash \exists a.\phi \) as required. \( \square \)

Definition 7.6. Suppose \( \phi \in \text{Pred} \) is a predicate and \( \text{minlev}(\phi) \geq 2 \) and \( S_\exists \text{consts}(\phi) \) (Notation 7.4). Then define

\[ \phi_\downarrow \]

to be that predicate obtained by replacing every variable \( a \) in \( \phi \) with \( \vartheta^{-1}(a) \) (Definition 2.5), and every constant \( x \) in \( \phi \) with \( x_\downarrow \) (Definition 6.4). An inductive definition would be routine to write out.

Corollary 7.7. Suppose \( \phi \in \text{CPred} \) is a closed predicate and \( \text{minlev}(\phi) \geq 2 \) and suppose \( k < \omega \) and \( S_k \text{consts}(\phi) \). Then

\[ k \vdash 2 \phi \text{ if and only if } k \vdash 1 \phi_\downarrow. \]

Proof. By induction on \( \phi \). We consider the cases of \( \in \) and \( = \) and \( \exists \) and use Figure 8:

— The case of \( x \in \chi \). From Lemma 6.10(1).

— The case of \( x = x' \). From Lemma 6.10(2).

— The case of \( \exists a.\phi \). Write \( a' = \vartheta^{-1}(a) \) and \( \phi' = \phi_\downarrow \). We consider two implications:

— Suppose \( k \vdash 2 \exists a.\phi \), so that \( k+1 \vdash 2 \phi[a:=x] \) for some \( x \in V_{\text{level}(a)} \) with \( S_{k+1} \tau x \). Write \( x' = x_\downarrow \) and note by Lemma 6.30(1) that \( x' \in V_{\text{level}(a')} \) and \( T_k \tau x' \).

It follows using the inductive hypothesis that \( k+1 \vdash 1 \phi'[a':=x'] \), and so \( k \vdash 1 \exists a'.\phi' \) as required.

— Now suppose \( k \vdash 1 \exists a'.\phi' \), so that \( k \vdash 1 \phi'[a':=x'] \) for some \( x' \in V_{\text{level}(a')} \) with \( T_k \tau x' \).

By Theorem 6.35 \( x' = x_\uparrow \downarrow \). Using the inductive hypothesis we deduce \( k \vdash 2 \exists a.\phi \). \( \square \)

7.3. The proof

Corollary 7.8 (Typical Ambiguity). Suppose \( \phi \in \text{CPred} \) is a closed predicate without constants and suppose \( \text{minlev}(\phi) \geq 2 \). Then

\[ \vdash \phi \iff \vartheta^{-1} \cdot \phi. \]

Proof. It is a fact of Definitions 7.6 and 2.6 that if \( \phi \) mentions no constants then \( \phi_\downarrow = \vartheta^{-1} \cdot \phi \).

Suppose \( \vdash \phi \). By Proposition 7.5 (since \( T_0 \text{consts}(\phi) \) trivially) \( 0 \vdash 2 \phi \). By Corollary 7.7 \( 0 \vdash 1 \vartheta^{-1} \cdot \phi \). By Proposition 7.5 again \( \vdash \phi_\downarrow \).

The reverse implication is identical. \( \square \)

8. SOUNDNESS, AND CONSISTENCY OF TST+

Theorem 8.1. Suppose \( \phi \in \text{CPred} \).

(1) If \( \vdash \phi \) is derivable using the rules of Figures 2 and 3 then \( \vdash \phi \).

(2) As a corollary, TST+ is consistent: \( \vdash \bot \) cannot be derived using the rules of Figure 2.

Proof. (1) We reason by induction on a derivation using the rules in Figure 2. Most cases are immediate by properties of sets:
— (instantiation) uses Corollary 4.9;
— (extensionality) uses Lemma 4.11; and
— (Leibniz) uses Lemma 4.10.

Furthermore:
— (comprehension) uses Proposition 4.19;
— (TA) uses Corollary 7.8; and
the other axioms are routine.

(2) From part 1 of this result, noting that $\not\models \bot$.  

**Remark 8.2 (Notes on the meta-theory).** We briefly discuss the strength of the theory required to express this proof:

(1) We assume GCH the generalised continuum hypothesis in Definition 3.8 and used it (via Lemma 3.7) to prove Theorem 8.1. So Theorem 8.1 in full states that

$$GCH \Rightarrow \text{Con}(\text{TST}+)$$

However the assertion ‘$\text{Con}(\text{TST}+)’$ is *arithmetical* and it follows using Schonfield’s absoluteness theorem [Sho61] that we may drop the assumption of GCH. We noted [Spe62] that NF is equiconsistent with TST+. Thus the implication above is sufficient to conclude

$$\text{Con}(\text{NF})$$

as required.

(2) In fact we use Choice in this paper too. As for GCH, because $\text{Con}(\text{NF})$ is arithmetical this can be eliminated in the sense that ‘$\text{AC}$ implies $\text{Con}(\text{NF})’$ implies ‘$\text{Con}(\text{NF})’’.

(3) We assume the existence of a set of size $\beth(\omega)$ in Definition 4.1. This is to be expected, by consistency strength arguments.

(4) We do not seem to have used Replacement in this proof.

**REFERENCES**


