One quantifier for forall and exists and generic elements,

or . . .

unknowns are data

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November 24, King's College London

Predicate logic, $\forall$ and $\exists$

Classical predicate logic satisfies the following equivalences:

$$\neg\forall x.\,\neg P \iff \exists x.\,P \qquad \neg\exists x.\,\neg P \iff \forall x.\,P.$$

This suggests that there is actually just *one* quantifier, with the power of both $\forall$ and $\exists$, sitting in between them.

Our mission: to discover this quantifier and retrieve its proof-theory.

Syntax of terms and predicates

Assume a set of variable symbols but call them atoms and write them $a, b, c, n \in \mathbb{A}$. Assume some language of terms; the logic is parametric over it. Write unknown terms s, t, and so on, and assume they are untyped. Write $t[a \mapsto t']$ for the usual substitution on terms. Write At for the set of atoms in t. Write $a \in t$ for $a \in At$ and $a \notin t$ for $a \notin At$.

Write $s \equiv t$ when s and t are syntactically identical terms, and $s \not\equiv t$ otherwise.

Assume predicate constant symbols $p, q, r \ldots \in \mathbb{P}$, each with an arity n. We assume distinguished constants:

1. $=$ has arity two and we call it equality.

2. $\mathbf{at}$ has arity one and we call it is an atom.

Write $p(ts)$ for a predicate constant applied to a list of n terms where n is the arity of p.

Define propositions or formulae by:

$$P ::= p(ts) \mid P \wedge P \mid P \vee P \mid P \Rightarrow P \mid \top \mid \bot \mid n\#P \mid \mathbf{at}\, a.$$

Write $\neg P$ for $P \Rightarrow \bot$ and $t \neq t'$ for $\neg(t = t')$. Write $P \equiv Q$ when P and Q are equal up to α-equivalence. n is bound in $n\#P$. Nothing else binds.

We write AP for the atoms occurring in P:

$$A(P \wedge Q) = AP \cup AQ \quad \cdots \quad A\bot = \emptyset$$
$$A(\mathbf{at}\, a) = \{a\} \quad A(n\#P) = AP \setminus \{n\}$$

Write $a \in P$ for $a \in AP$ and $a \notin P$ for $a \notin AP$.

Intuitions

- Read $n\#P$ as 'n fresh for P'. It means "P is independent of the value of n", or formally $\forall n.\ P \vee \forall n.\ \neg P$. We shall see how to express $\forall$ using $\#$.

  Theorems include $a\#(a = a)$ and $a\#a \neq a$. Note $a\#P$ yet $a \in P$.

- Read $\mathbf{at}\, a$ as 'a *is* an atom'. So $\neg(\mathbf{at}\,\langle a, a\rangle)$ is a theorem. Yet $(\mathbf{at}\, a)$ is not a theorem, since we want the substitution lemma: if P is provable then $\vdash\ P\sigma$ is provable.

$$\frac{}{\Gamma, P \vdash P, \Delta}\ (Ax)$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ (\bot L) \qquad \frac{}{\Gamma \vdash \top, \Delta}\ (\top R)$$

$$\frac{\Gamma, \Delta}{\Gamma \vdash \bot, \Delta}\ (\bot R) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}\ (\top L)$$

$$\frac{\Gamma \vdash P, \Delta\ \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}\ (\wedge R) \qquad \frac{\Gamma, P, Q \vdash C, \Delta}{\Gamma, P \wedge Q \vdash C, \Delta}\ (\wedge L)$$

$$\frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}\ (\vee R) \qquad \frac{\Gamma, P \vdash C, \Delta\ \Gamma, Q \vdash C, \Delta}{\Gamma, P \vee Q \vdash C, \Delta}\ (\vee L)$$

I'm tired of all these 'Γ's and 'Δ's . . .

Rules (routine part)

…please take them as read.

$$\frac{P \vdash Q}{\vdash P \Rightarrow Q}(\supset R) \quad \frac{\vdash P \quad Q \vdash}{P \Rightarrow Q \vdash}(\supset L)$$

$$\frac{\vdash P \quad P \vdash Q}{\vdash Q}(Cut) \quad \frac{P, P \vdash}{P \vdash}(Lct) \quad \frac{\vdash C, C}{\vdash C}(Rct)$$

$$\frac{}{\vdash t = t}(=R) \quad \frac{s' = s, P[a \mapsto s'] \vdash}{s' = s, P[a \mapsto s] \vdash}(=Sub)$$

Rules (novel)

$$\frac{P[n\mapsto t]\ \vdash\ P[n\mapsto b] \quad P[n\mapsto b]\ \vdash\ P[n\mapsto t]}{\vdash\ n\#P}\,(\#R)\; b \notin \Gamma, \Delta, P$$

$$\frac{\vdash\ P[n\mapsto t] \quad P[n\mapsto t']\ \vdash}{n\#P\ \vdash}\,(\#L)$$

$$\frac{t \text{ not an atom}}{\mathbf{at}\, t\ \vdash}\,(\mathbf{at}\, L)$$

$$\frac{\mathbf{at}\, a\ \vdash}{\vdash}\,(FreshL)\quad a \notin \Gamma, \Delta \qquad \frac{\vdash\ \mathbf{at}\, a}{\vdash}\,(FreshR)\quad a \notin \Gamma, \Delta$$

Example derivations

It is convenient to introduce faux derivation rules as shorthand:

$$\frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}(\neg L) \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}(\neg R)$$

These are standard derivation rules for $\neg$ when it is a predicate-former. For us $\neg P$ defined as shorthand for $P \Rightarrow \bot$, so we use the derivation rules above as shorthand for the following, just to save space:

$$\frac{\Gamma \vdash P, \Delta \qquad \dfrac{}{\Gamma, \bot \vdash C}(\bot L)}{\Gamma, \neg P \vdash \Delta}(\Rightarrow L) \qquad \frac{\dfrac{\Gamma, P \vdash \Delta}{\Gamma, P \vdash \bot, \Delta}(\bot R)}{\Gamma \vdash \neg P, \Delta}(\Rightarrow R)$$

$$\dfrac{\dfrac{}{\mathbf{at}\,\langle a,a\rangle\ \vdash}\,(\mathbf{at}\,L)}{\vdash\ \neg(\mathbf{at}\,\langle a,a\rangle)}\,(\neg R)$$

$$\dfrac{\dfrac{}{a=a\ \vdash\ b=b}\,(=R)\qquad\dfrac{}{b=b\ \vdash\ a=a}\,(=R)}{\vdash\ n\#(n=n)}\,(\#R)$$

$$\dfrac{\dfrac{}{a\neq a\ \vdash\ b\neq b}\,(\neg L),(\neg R),(=R)\qquad\dfrac{}{b\neq b\ \vdash\ a\neq a}\,(\neg L),(\neg R),(=R)}{\vdash\ n\#(n\neq n)}\,(\#R)$$

Generic elements

Define $\text{И}n.\,P \equiv n\#(\mathbf{at}\,n \Rightarrow P)$.

With this definition, the following left- and right-introduction rules are admissible:

$$\frac{\mathbf{at}\,a, P[n\mapsto a] \;\vdash}{\text{И}n.\,P \;\vdash}\,(\text{И}L) \qquad \frac{\mathbf{at}\,a \;\vdash\; P[n\mapsto a]}{\vdash\; \text{И}n.\,P}\,(\text{И}R)$$

where in both rules there is the side-condition $a \notin \Gamma, \Delta, P$.

Here are the derivations:

Admissible rule for И

$$
\dfrac{
  \dfrac{
    \dfrac{}{\mathbf{at}\,a,\mathbf{at}\,b \;\vdash\; P[n\mapsto b],\mathbf{at}\,b}\,(Ax)
  }{
    \mathbf{at}\,a \;\vdash\; \mathbf{at}\,b \Rightarrow P[n\mapsto b],\ \mathbf{at}\,b
  }\,(\Rightarrow R)
  \qquad
  \dfrac{
    \dfrac{}{\mathbf{at}\,a \;\vdash\; \mathbf{at}\,a,\mathbf{at}\,b}\,(Ax)
    \qquad
    \dfrac{\Pi[n\mapsto a]}{\mathbf{at}\,a, P[n\mapsto a] \;\vdash\; \mathbf{at}\,b}
  }{
    \mathbf{at}\,a,\ \mathbf{at}\,a \Rightarrow P[n\mapsto a] \;\vdash\; \mathbf{at}\,b
  }\,(\Rightarrow L)
}{
  \dfrac{
    \mathbf{at}\,a,\ n\#(\mathbf{at}\,n \Rightarrow P) \;\vdash\; \mathbf{at}\,b
  }{
    n\#(\mathbf{at}\,n \Rightarrow P) \;\vdash
  }\,(FreshL),(FreshR)
}\,(\#L)
$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{\mathbf{at}\,a,\mathbf{at}\,b \;\vdash\; \mathbf{at}\,a, P[n\mapsto b]}
    \qquad
    \dfrac{\Pi[n\mapsto b]}{\mathbf{at}\,a,\mathbf{at}\,b, P \;\vdash\; P[n\mapsto b]}
  }{
    \mathbf{at}\,a,\mathbf{at}\,a \Rightarrow P[n\mapsto a] \;\vdash\; \mathbf{at}\,b \Rightarrow P[n\mapsto b]
  }
  \qquad
  \dfrac{
    \dfrac{\Pi[n\mapsto a]}{\mathbf{at}\,a, P[n\mapsto b] \;\vdash\; P[n\mapsto a]}
  }{
    \mathbf{at}\,a,\mathbf{at}\,b \Rightarrow P[n\mapsto b] \;\vdash\; \mathbf{at}\,a \Rightarrow P[n\mapsto a]
  }
}{
  \vdash\; n\#(\mathbf{at}\,n \Rightarrow P)
}\,(FreshL),(\#R)
$$

$(FreshL)$ introduces an atom a which is known to be an atom, and $(FreshR)$ introduces an atom which is known *not* to be an atom!

Generic elements (cont.)

It is not hard to prove that $\neg(\text{И}n.\ P) \iff \text{И}n.\ \neg P$. Thus, P holds of a generic element if and only it holds of any other generic element.

Universal and existential quantification

$$\forall n.\, P \equiv (n \# P) \wedge (\text{И} n.\, P) \qquad \exists n.\, P \equiv (n \# P) \Rightarrow (\text{И} n.\, P)$$

So $\forall n.\, P$ holds when (a) P does not depend on n and (b) P is true of some n (use И to choose it).

$\exists n.\, P$ holds when either (a) P is true of some element and false of another or (b) P does not depend on n and is true of some element (use И to choose it).

Universal quantification rules are admissible

Given a derivation Π of $\Gamma \vdash P[n{\mapsto}a], \Delta$ for $a \notin \Gamma, \Delta, P$ we extend it to a derivation of $\Gamma \vdash \forall n.\, P, \Delta$:

$$\dfrac{\dfrac{\dfrac{\Pi}{\textbf{at}\ a \vdash P[n{\mapsto}a]}}{\vdash \text{И}n.\, P}\,(\text{И}R) \qquad \dfrac{\dfrac{\dfrac{\Pi[n{\mapsto}b]}{\textbf{at}\ a, P[n{\mapsto}a] \vdash P[n{\mapsto}b]} \qquad \dfrac{\Pi}{\textbf{at}\ a, P[n{\mapsto}b] \vdash P[n{\mapsto}a]}}{\textbf{at}\ a \vdash n\#P}}{\vdash n\#P}\,(FreshL), (\#R)}{\vdash n\#P \wedge \text{И}n.\, P}\,(\wedge R)$$

Given a derivation Π of $\Gamma, P[n{\mapsto}t] \vdash \Delta$ we extend it to a derivation of $\Gamma, \forall n.\, P \vdash \Delta$:

$$\dfrac{\dfrac{\dfrac{}{\textbf{at}\ a, P[n{\mapsto}a] \vdash P[n{\mapsto}a]}\,(Ax) \qquad \dfrac{\Pi}{\textbf{at}\ a, P[n{\mapsto}a], P[n{\mapsto}t] \vdash}}{\textbf{at}\ a, n\#P, P[n{\mapsto}a] \vdash}\,(\#L)}{n\#P \wedge \text{И}n.\, P \vdash}\,(\wedge L), (\text{И}L)$$

Given a derivation Π of $\Gamma \vdash P[n \mapsto t], \Delta$ we extend it to a derivation of $\Gamma \vdash \exists n.\ P, \Delta$:

$$\dfrac{\dfrac{\dfrac{}{\textbf{at}\ a, P[n\mapsto a] \vdash P[n\mapsto a]}(Ax) \qquad \dfrac{\Pi}{\textbf{at}\ a, P[n\mapsto a] \vdash P[n\mapsto t]}}{\textbf{at}\ a, n\#P \vdash P[n\mapsto a]}(\#L)}{\vdash n\#P \Rightarrow \text{И}n.\ P}(\Rightarrow R), (\text{И}R)$$

Given a derivation Π of $\Gamma, P[n\mapsto a] \vdash \Delta$ for $a \notin \Gamma, \Delta, P$ we extend it to a derivation of $\Gamma, \exists n.\ P \vdash \Delta$:

$$\dfrac{\dfrac{\dfrac{\Pi}{\textbf{at}\ a, P[n\mapsto a] \vdash}}{\text{И}n.\ P \vdash}(\text{И}L) \qquad \dfrac{\dfrac{\Pi}{\textbf{at}\ a, P[n\mapsto a] \vdash P[n\mapsto b]} \qquad \dfrac{\Pi[n\mapsto b]}{\textbf{at}\ a, P[n\mapsto b] \vdash P[n\mapsto a]}}{\vdash n\#P}(FreshL), (\#R)}{n\#P \Rightarrow \text{И}n.\ P \vdash}(\Rightarrow L)$$

Write $Term$ for terms.

If f is partial from A to Y write $dom(f)$ for the $a \in A$ such that fa is defined. If $dom(f)$ is finite say f is finite. Write $\mathbb{A} \rightarrow_{fin} X$ for the set of finite partial functions σ from $\mathbb{A}$ to X.

Write $\mathbb{B}$ for some two-element set $\{\mathbf{true}, \mathbf{false}\}$. We interpret it as a set of truth values.

Write $S, T \in \mathcal{S}$ for the set of infinite trees coinductively defined by

(1) $$\mathcal{S} \cong \mathbb{B} \times ((\mathbb{A} \rightarrow_{fin} Term) \rightarrow \mathcal{S}).$$

S is $(b \in \mathbb{B}, f \in (\mathbb{A} \rightarrow_{fin} Term) \rightarrow \mathcal{S})$. View S as a tree with nodes labelled by truth values and with infinite branching, one edge one for each partial valuation σ. Write $\pi_1 S$ for b and $\pi_2 S$ for f.

We are lax and write $s\sigma$ for $(\pi_2 s)\sigma$. Write σa for $\sigma(a)$ when $a \in dom(\sigma)$ and a otherwise.

Semantics

$\mathbf{at}\ a$ is the tree with nodes under an edge labelled σ, labelled by $\mathbf{true}$ precisely when σa is an atom (variable symbol).

$n\#P$ has nodes under an edge labelled σ, labelled $\mathbf{true}$ when $P\sigma[n\mapsto t]$ has label consistently true or consistently false for all t, where n avoids capture by σ.

$\text{И}n.\ P$ has node under an edge labelled σ, labelled $\mathbf{true}$ when $P\sigma$ is labelled true, where n avoids capture.

$\forall n.\ P$ has node under an edge labelled σ, labelled ...

Properties of the logic

Work-in-progress, but I believe I have the following:

Substitution Lemma: If $\Gamma \vdash \Delta$ then $\Gamma\sigma \vdash \Delta\sigma$.

Cut Elimination: Cut is an admissible rule in the system without it.

Soundness and completeness: The logic is sound and complete with respect to a slightly refined version of the semantics described.

Future work and applications

- Abstract algebra. Semantics give a notion similar to cylindric algebras, but for infinitely many variables. We obtain an algebra which is to first-order predicate logic as boolean algebra is to propositional logic.
- Foundations. I suspect that underlying this work is an 'untyped' alternative to category theory. Composition of arrows is replaced by substitution. (Connections here to 'Reasoning with arbitrary objects' by Kit Fine.)
- Programming. $=$-free fragment might make an interesting type system. Also, is there a λ-calculus with И and substitution, instead of λ and application. Connections to category theory.
- Itself. Surely this is just a cool logic to have.

Future work and applications

Unknowns are data. They are generated by $\mathsf{И}$ and assertions can be made about them using $\mathbf{at}$, $\#$, and $=$. This logic realises part of these ideas in a formal framework. More to follow.