Incomplete objects using the NEW calculus of contexts

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This talk

My understanding is that I have ten or fifteen minutes.

So I’ll just put some stuff on the projector and talk about it.

Incomplete objects and instantiation

Capture-avoiding substitution avoids capture:

$$(\lambda x.y)[y{\mapsto}x] \rightsquigarrow \lambda x'.x$$

Instantiation (my terminology) does not:

$$(\lambda x.Y)[Y{\mapsto}x] \rightsquigarrow \lambda x.x$$

Say Y is stronger than x (because it can capture it).

Incomplete objects and instantiation

The NEW calculus of contexts is a λ-calculus with a hierarchy of variables such that:

- Substitution for a weak variable (weak substitution) avoids capture relative to variables of the same strength or stronger.
- Substitution for a strong variable (strong substitution) instantiates relative to strictly weaker variables.

Incomplete objects

This is a model of incomplete objects.

For example $\lambda x.T$, where x has level 1 and T has level 2, models an incomplete term.

A strong substitution for T can arrive and turn T into any term — also one mentioning x. In this sense, $\lambda x.T$ is ‘incomplete’.

HOL in the NEWcc

The NEWcc admits a type system.

We can hypothesise types $\mathbb{B}$ of propositions and $\mathbb{T}$ of terms. We can generate types by:

$$\tau ::= \mathbb{B} \quad | \quad \mathbb{T} \quad | \quad \tau \rightarrow \tau$$

This is standard, e.g. Isabelle/FOL has a type of first-order logic propositions o, COQ has $Prop$, HOL also has o, and so on.

Constants

We can hypothesise constant symbols:

- $\bot : \mathbb{B}$.
- $\Rightarrow : \mathbb{B} \longrightarrow \mathbb{B} \longrightarrow \mathbb{B}$.
- $\forall : (\mathbb{T} \longrightarrow \mathbb{B}) \longrightarrow \mathbb{B}$.
- $=_\tau : \tau \longrightarrow \tau \longrightarrow \mathbb{B}$.

Some notation

- Import usual sugar from classical logic.
- Let P, Q, R be unknowns of level 2 of type $\mathbb{B}$.
- Let x, y, z be unknowns of level 1 of type $\mathbb{T}$.
- Let T, U, V be unknowns of level 2 of type $\mathbb{T}$.
- Write $\forall a.t$ for $\forall \lambda a.t$.

Axioms

We can hypothesise axioms:

- $\text{И}x.\ \forall P.(P \Rightarrow \forall x.P)$.
- $\text{И}x.\ \forall P.\forall y.((\forall x.P) \Rightarrow P[x \mapsto y])$.
- $\forall P.\text{И}x.\ \forall Q.\Big((\forall x.(P \Rightarrow Q)) \Rightarrow (P \Rightarrow \forall x.Q)\Big)$.

И is the Gabbay-Pitts NEW quantifier. It generates a fresh variable symbol (unlike $\forall$, which generates an arbitrary element). The order of quantifiers enforces what is fresh for what, or in more standard terminology, enforces dependencies.

Theorems

We can hypothesise some theorems too:

- $\forall x.\exists y.(x =_\tau y)$.
- $\exists T.\forall x.(x =_\tau T)$ (Skolemisation, without functions).
- $\forall z.\forall z'.\text{И}x.\, \forall P.(z =_\tau z') \Rightarrow (P[x\mapsto z] \Leftrightarrow P[x\mapsto z'])$.

That is, I hypothesise that these might be theorems.

Terms

x is a term of type $\mathbb{T}$.

T is also a term of type $\mathbb{T}$.

However, $\lambda y.x$ is boring (the constant function mapping any element of $\mathbb{T}$ to x).

$\lambda y.T$ is rather interesting because T might depend on y.

So is $\lambda T.\lambda y.T$; again, T may depend on y — even though it comes first.

$\lambda T.\text{И}y.\ \lambda y.T$ is boring again. Independence is enforced.

So in the NEWcc there may be nontrivially 'incomplete' elements, like T which depends on an argument we have not yet received.

Conclusions

NEWcc level 2 variables represent ‘incomplete terms of level 1’.

These incomplete terms can be λ-abstracted over.

Subsequent substitution for the level 2 variable need not avoid capture, if the abstractions are at level 1.

This internalises meta-level reasoning.

There are levels 3, 4, and so on, which give further power to express subtle interactions of capture-avoidance and instantiation. (Any poset will do for levels.)

Conclusions

We just discussed validity ($\mathbb{B}$), in Isabelle style — Isabelle by default has no proof-terms.

I suspect that in a richer structure with a type of proofs rather than a type of propositions, more is possible; ‘incomplete proofs’, or strategies.