Nominal Logic

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13 November 2006

Thanks to Mark Price and Alessio Gugliemi

Nominal logic

Nominal logic is an extension of first-order logic for reasoning with names.

Formulae of first-order logic

Formulae are inductively defined by:

$$\phi ::= \bot \mid \phi \Rightarrow \phi \mid \forall x.\phi \mid t = t.$$

We use standard classical logic sugar:

$$\neg\phi \text{ is } \phi \Rightarrow \bot \qquad \top \text{ is } \neg\bot$$

$$\phi \wedge \psi \text{ is } \neg(\phi \Rightarrow \neg\psi) \qquad \phi \vee \psi \text{ is } \neg\phi \Rightarrow \psi$$

$$\phi \Leftrightarrow \psi \text{ is } (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \qquad \exists x.\phi \text{ is } \neg\forall x.\neg\phi$$

Derivation rules of first-order logic

$$\frac{}{\phi,\ \Phi \vdash \Psi,\ \phi}\,(\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash \Psi}\,(\bot\mathbf{L})$$

$$\frac{\Phi \vdash \Psi,\ \phi \quad \psi,\ \Phi \vdash \Psi}{\phi \Rightarrow \psi,\ \Phi \vdash \Psi}\,(\Rightarrow\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \phi \Rightarrow \psi}\,(\Rightarrow\mathbf{R})$$

$$\frac{\phi[\![x{\mapsto}t]\!],\ \Phi \vdash \Psi}{\forall x.\phi,\ \Phi \vdash \Psi}\,(\forall\mathbf{L}) \qquad \frac{\Phi \vdash \Psi,\ \phi}{\Phi \vdash \Psi,\ \forall x.\phi}\,(\forall\mathbf{R}) \qquad (a \text{ fresh})$$

$$\frac{\phi[\![x{\mapsto}t']\!],\ \Phi \vdash \Psi}{t' = t,\ \phi[\![x{\mapsto}t]\!],\ \Phi \vdash \Psi}\,(=\mathbf{L}) \qquad \frac{}{\Phi \vdash \Psi,\ t = t}\,(=\mathbf{R})$$

Data

A slogan:

Names are data.

Compare with another slogan: Numbers are data.

Data is represented by terms, e.g. to represent numbers assume term-formers $\mathsf{0}$ and succ. We may also wish to associate some axioms to this data.

For example:

$$\forall x.\forall y.(\mathsf{succ}(x) = \mathsf{succ}(y) \Rightarrow x = y) \qquad \mathsf{0} \neq \mathsf{succ}(x)$$

$$\phi[\![x{\mapsto}\mathsf{0}]\!] \Rightarrow \forall x.(\phi \Rightarrow \phi[\![x{\mapsto}\mathsf{succ}(x)]\!]) \Rightarrow \forall x.\phi.$$

(The last ‘axiom’ is actually an axiom-scheme.)

What kind of data is a name?

Good question! What kind of data is a name?

There are no term-formers: (for us) names are atomic.

Introduce a type $\mathbb{A}$ of names. The only terms of type $\mathbb{A}$ are variable symbols. Write variable symbols of type $\mathbb{A}$ as $a, b, c, \ldots$

What are the properties of names?

Names can be interchanged

Property 1: Equivariance.

$$\phi(x_1, \ldots, x_n) \Leftrightarrow \phi((a\ b)x_1, \ldots, (a\ b)x_n).$$

Property 2: Finite support.

$$\text{И}b.\text{И}a.(a\ b)x = x.$$

We silently universally quantify free variables in axioms above; $\phi(x_1, \ldots, x_n)$ denotes a predicate with free variables at most $\{x_1, \ldots, x_n\}$.

That's it! Thank you very much for your attention.

It was lovely talking to you. Shall we go to coffee now?

Names can be interchanged

What's that?

You want to know what $(a\ b)x$ and $\reflectbox{N}$ are?

Did I forget to mention?

The principle of Equivariance

Actually, names do have term-formers. Swapping $(a\ b)t$ swaps a and b in t.

E.g. if t is the text `abracadabra` of a program and names are ASCII `a`, `b`, ..., then:

$$(\mathtt{a}\ \mathtt{b})\mathtt{abracadabra} = \mathtt{barbcbdbarb}.$$

Equivariance states that truth is invariant (symmetric) up to swapping names:

$$\phi(x_1, \ldots, x_n) \Leftrightarrow \phi((a\ b)x_1, \ldots, (a\ b)x_n).$$

Examples

Only one, really:

$$\mathtt{a} = \mathtt{b} \Leftrightarrow \mathtt{b} = \mathtt{a} \qquad \mathtt{a} = \mathtt{c} \Leftrightarrow \mathtt{b} = \mathtt{c} \qquad \mathtt{c} = \mathtt{a} \Leftrightarrow \mathtt{c} = \mathtt{b}$$

All other predicates are built up from this using $\Rightarrow$, $\bot$, and $\forall$.

Suppose f is a term-former. Then

$$\mathsf{f}(x_1, \ldots, x_n) = z \Leftrightarrow \mathsf{f}((a\ b)x_1, \ldots, (a\ b)x_n) = (a\ b)z$$

therefore we have equivariance for terms:

Lemma: For any term-former f in the language of terms,

$$(a\ b)\mathsf{f}(x_1, \ldots, x_n) = \mathsf{f}((a\ b)x_1, \ldots, (a\ b)x_n).$$

The NEW quantifier N

The intuitive meaning of $\mathsf{N}a.\phi$ is:

$$\exists S \text{ a finite set of atoms.} \forall b \notin S.\phi[\![a \mapsto b]\!] \Leftrightarrow \phi.$$

So $\mathsf{N}a.\phi$ means 'ϕ holds for all but finitely many a' or 'ϕ holds mostly'.

Finite support

So $\text{И}b.\text{И}a.(b\ a)x = x$ means 'for most a and most b, $(a\ b)x = x$'.

For example if $x = \texttt{abracadabra}$ then:

- Most atoms a are not in $\{\texttt{a}, \texttt{b}, \texttt{r}, \texttt{c}, \texttt{d}\}$.
- Most atoms b are not in $\{\texttt{a}, \texttt{b}, \texttt{r}, \texttt{c}, \texttt{d}\}$.

Therefore

$$\text{И}b.\text{И}a.\big((b\ a)\texttt{abracadabra} = \texttt{abracadabra}\big).$$

Finite support

Suppose that $x = \{\mathtt{a}, \mathtt{c}, \mathtt{e}, \mathtt{g}, \mathtt{i}, \mathtt{k}, \mathtt{m}, \mathtt{o}, \mathtt{q}, \mathtt{s}, \mathtt{u}, \mathtt{w}, \mathtt{y}, \ldots\}$ (x mentions 'every other atom').

Then $\text{И}b.\text{И}a.(b\ a)x = x$ does not hold. So the set of 'every other atom' is not permitted.

However the set $y = \{\mathtt{a}, \mathtt{b}\}$ is permitted, because if you swap two atoms not in y then you get y, and most atoms are not in y.

Also $z = \{\mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \ldots\}$ is permitted, because if you swap two atoms in z then you get z, and most atoms are in z.

Freshness

Write $a\#x$ when $\text{И}b.(b\,a)x = x$. Read this as 'a is fresh for x'. Write $\mathsf{supp}(x) = \{a \mid \neg(a\#x)\}$.

So $\text{И}b.\text{И}a.(b\,a)x = x$ means $\text{И}a.a\#x$ means $\mathsf{supp}(x)$ is finite.

Lemma: If $a\#x$ and $b\#x$ then $(b\,a)x = x$.

Proof: $(c\,a)x = x$ for most c. $(c\,b)x = x$ for most c. Choose some common c such that $(c\,a)x = x$ and $(c\,b)x = x$. Then

$$(a\,b)x = (a\,b)(c\,a)(c\,a)x = (c\,a)(c\,b)(c\,a)x = x.$$

A corollary

Theorem: $\text{И}a.\phi \Leftrightarrow \neg\text{И}a.\neg\phi$.

Proof: Suppose $\text{И}a.\phi$. Then clearly $\neg\text{И}a.\neg\phi$.

Suppose $\neg\text{И}a.\neg\phi$; it is not the case that most $\mathtt{a}$ satisfy $\phi[\![a{\mapsto}\mathtt{a}]\!]$.

(Abuse of notation: $\mathtt{a}$ is not in the language of terms.)

So for any finite set of atoms there is some $\mathtt{a}$ not in that set such that $\phi[\![a{\mapsto}\mathtt{a}]\!]$.

As finite set take $\mathsf{supp}(x_1) \cup \ldots \cup \mathsf{supp}(x_n)$. Here the $x_1, \ldots, x_n$ are the free variables of ϕ other than a.

So $\phi[\![a{\mapsto}\mathtt{a}]\!]$.

A corollary

By equivariance for any b

$$\phi[\![a \mapsto \mathtt{a}]\!] \Leftrightarrow (\mathtt{b}\ \mathtt{a})(\phi[\![a \mapsto \mathtt{a}]\!]).$$

Now $(\mathtt{b}\ \mathtt{a})\mathtt{a} = \mathtt{b}$. By our lemma if $\mathtt{b} \notin \mathsf{supp}(x_1) \cup \ldots \cup \mathsf{supp}(x_n)$ then

$$\phi[\![a \mapsto \mathtt{a}]\!] \Leftrightarrow \phi[\![a \mapsto \mathtt{b}]\!].$$

But most b are not in $\mathsf{supp}(x_1) \cup \ldots \cup \mathsf{supp}(x_n)$. So

$$\text{И}a.\phi.$$

Properties of N

We conclude $\mathsf{N}a.\phi \Leftrightarrow \neg\mathsf{N}a.\neg\phi$. N is self-dual.

Properties of N are:

$$\mathsf{N}a.(\phi \wedge \psi) \Leftrightarrow (\mathsf{N}a.\phi \wedge \mathsf{N}a.\psi)$$

$$\mathsf{N}a.(\phi \vee \psi) \Leftrightarrow (\mathsf{N}a.\phi \vee \mathsf{N}a.\psi)$$

$$\mathsf{N}a.(\phi \Rightarrow \psi) \Leftrightarrow (\mathsf{N}a.\phi \Rightarrow \mathsf{N}a.\psi)$$

What do you want next?

Discussion of abstract syntax with binding in Nominal Logic?

Discussion of sequent rules for И?

Discussion of possible extensions?