Substitution for Fraenkel-Mostowski Foundations

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Computer science

We work in the foundations of computer science; we make logics, set theories, and similar.

Computer science continues to throw up new problems.

Some of these problems are connected with the mathematical foundations which we use.

Some of the foundational implications link to issues of arbitrariness that philosophers of mathematics have thought about for centuries.

Computer science

What is the difference between the following two terms/programs?

1. $\lambda x.fx$
2. $\lambda y.fy$

In a sense, nothing, both correspond to the function f. But in another sense, there is a difference. To a computer, which crunches symbols, they appear to be different operations.

Computer science and Philosophy

What is the difference between the following two (partial) assertions?

1. Let x be a number, we may apply f to it to yield...
2. Let y be a number, we may apply f to it to yield...

A natural answer: semantically nothing, but there are syntactic differences.

But the difference between x and y is not like that of ‘groundhog’ and ‘woodchuck’, for we not entitled to assert $x = y$.

Arbitrary Objects

A natural attempt to solve this issue is to theorise that x and y refer to distinct arbitrary objects. In the case above, arbitrary numbers.

This suggests a simple semantic account of ‘general’ assertions. General assertions are like particular assertions, except that the particulars happen to be ‘general’ objects! Similarly with knowledge and understanding.

Problems with arbitrary objects

Arbitrary objects have peculiar properties. For example an arbitrary pig

> . . . has just those properties that every pig has. Since not every pig is pink, grey, or any other color, the universally generic pig is not of any color. (Yet neither is he colorless, since not every-indeed not any-pig is colorless.) Nor is he(?) male or female (or neuter), since not every pig is any one of these. He is, however, a pig and an animal, and he grunts; for every pig is a pig and an animal, and grunts. [Lewis, *General Semantics*]

Fine on arbitrary objects

Kit Fine, in *Reasoning with arbitrary objects*, constructs an algebra of arbitrary objects which makes logical sense of Lewis's complaint.

But how do arbitrary objects relate to semantics?

1. Let x be a number, then Fx.
2. $v(Fx) = \top$ for every x-alternate valuation to v.

We can set up an algebra of arbitrary objects to make 1 and 2 equivalent. However, a foundational account from which the equivalence, and the algebra, just 'fall out', would be better.

For example, the algebra of propositions 'falls out' from the sets interpretation of propositional logic.

Not all arbitrary terms are universal

Consider a famous example:

- If a farmer has a donkey then he beats it.

The expression ‘a farmer’ has universal import. But what about:

- Suppose someone is next door, then they are either dead or very quiet.
- If I have a pound, then I’ll put it in the parking meter.

Existential arbitrary terms?

A mathematical example:

- Let X be a set, and f be a choice function on sets, if X has a member y then we may assume that $fX = y \ldots$ it follows that $A(y)$.

Does it even make sense to have an arbitrary-particular y?

A better solution?

We can use the techniques for expressing generality supplied to us by Frege and developed in modern logic. A sentence such as

- A positive integer is either odd or even.

should be read as either

- [For every x] if x is a positive integer, then x is either odd or even.

Where the expression ‘For every x’ is either implicit in the assertion, or to be added later.

This solution does not provide a direct semantics to arbitrary terms.

Semantics for the universal quantifier

The semantics for the universal quantifier are not as straightforward as one might think. We are all familiar with:

> $v(\forall x.Fx) = \top$ iff $v(Fx) = \top$ for every x-alternate valuation to v.

But this does not identify the universal quantifier with an operation on the content of Fx. Put another way:

> If the proposition A corresponds to a set $|A|$, what operation f does the universal quantifier correspond to so that $f|A| = |\forall x.A|$?

Tarski semantics does not provide an answer.

Infinite conjunction?

We might treat the universal quantifier as a kind of infinite intersection:

$$|\forall x.A| = \bigcap_{d\in D} |A[x \mapsto d]|.$$

But $[x \mapsto d]$, is not a semantic operation, it is not purely an operation on contents.

It partly syntactic as it makes explicit reference to variables.

Arbitrary objects and substituion

If we treat can treat substitution semantically, then we can create a semantics in which arbitrary terms refer to arbitrary objects, directly in the semantics.

The rôle of ‘valuation’ is played by substitution in the semantics.

We obtain a semantic model for quantifiers as operations involving substitutions in the semantics.

ZF set theory

If computer scientists think about foundations they usually think about Zermelo-Fraenkel (ZF) set theory.

The user's idea of ZF is:

- The empty collection is a set.
- If X is a set then the collection of subsets of X is a set.

Nice, easy idea.

$\{\}$ is a set. $\{\} \subseteq \{\}$ so $\{\{\}\}$ is a set. And so on.

A problem with this idea

There is no ‘abstract set’.

Consider an example: let X be an element. Then X has to be a particular set.

It is now impossible to use the fact that we never cared what X was.

Fraenkel-Mostowski set theory

Fraenkel-Mostowski set theory (FM). Developed in the 1930s to prove the independence of the Axiom of Choice from the other axioms of set theory. The user's idea is:

- The empty set is an element.
- Atoms (urelemente) $a, b, c, d, \ldots$ are elements. Write $\mathbb{A}$ for the set of all atoms.
- If X is an element then the set of finitely-supported subsets of X is an element.

I will discuss 'finitely-supported' shortly.

Atoms

Atoms are particular elements, but they model arbitrary elements.

We can make this mathematically precise:

If π is a bijection on atoms then we can permute them in any X:

$$\pi \cdot X = \{\pi \cdot x \mid x \in X\}$$

For example $\pi \cdot \{\{a\}, b\} = \{\{\pi(a)\}, \pi(b)\}$.

Equivariance

Note that:

- $x \in y$ if and only if $\pi \cdot x \in \pi \cdot y$ (since $\pi \cdot y = \{\pi \cdot x \mid x \in y\}$).
- $x = y$ if and only if $\pi \cdot x = \pi \cdot y$ (since π is a bijection on atoms).

It follows that for any predicate $\phi(x_1, \ldots, x_n)$ in the language of sets,

$$\phi(x_1, \ldots, x_n) \quad \text{if and only if} \quad \phi(\pi \cdot x_1, \ldots, \pi \cdot x_n).$$

This is equivariance: something that is true/false of $(x_1, \ldots, x_n)$ (and the atoms they contain) is just as true/false if we permute atoms.

Equivariance

Thus equivariance states: in the world of FM, atoms are arbitrary up to permuting them.

Atoms exist — they are elements. We have a, and b, and they are distinct elements.

But, addressing our earlier point, even if we choose one particular atom, anything we prove will be as true if we had made a different choice.

It remains to define a notion of substitution of atoms for sets. This allows us to instantiate an atom (which model arbitrary elements) to ... an arbitrary set.

Finite support

Kit Fine pointed out in his book on arbitrary objects that arbitrary objects need not be independent of each other. For example, 'n' and '$2 * n$' are two 'arbitrary numbers', but $2 * x$ is guaranteed to be precisely twice as large as n; they are connected.

FM sets achieves a similar effect using support.

Say that $S \subseteq \mathbb{A}$ supports x when: If $\pi(a) = a$ for all $a \in S$, then $\pi \cdot x = x$. For example:

- $\{a\}$ supports a.
- $\{a\}$ supports $\mathbb{A} \setminus \{a\} = \{b, c, d, \ldots\}$.

If $\pi(a) = a$ then
$\pi \cdot \{b, c, d, \ldots\} = \{\pi(b), \pi(c), \pi(d), \ldots\} = \{b, c, d, \ldots\}$.

Finite support

X is finitely supported when X has some finite supporting set.

For example $\{a\}$ and $\mathbb{A} \setminus \{a\}$ are finitely supported, by $\{a\}$.

$\mathbb{A} = \{a, b, c, d, \ldots\}$ is finitely supported by $\{\}$, since

$$\begin{aligned}\pi \cdot \{a, b, c, d, \ldots\} = \{\pi(a), \pi(b), \pi(c), \pi(d), \ldots\} \\ = \{a, b, c, d, \ldots\}.\end{aligned}$$

$\{\{a\}, \{\{a\}, \{b\}\}\}$ is finitely supported by $\{a, b\}$.

A set of 'every other atom' $\{a, c, e, g, \ldots\}$ is not finitely supported. No matter what finite set S we choose, we can permute atoms outside of S.

Finite support

Write $a\#x$ when x is supported by some S and $a \notin S$.

This gives a concrete model of the notion of dependence investigated algebraically by Kit Fine.

Think of $a\#x$ as 'x does not depend on a'.

Remember, this is not the same thing as 'a does not occur in x'. For example

- $a\#\mathbb{A}$ and $a \in \mathbb{A}$, and
- $b\#\{b, c, d, \ldots\}$ and $b \in \{b, c, d, \ldots\}$.

Substitution

If we can define a notion of substitution on FM sets, then this has philosophical implications.

It suggests that atoms are arbitrary in the sense that they can be instantiated to other elements.

Substitution

We belong to a technical field. The definition of substitution is technical — and quite subtle.

We hope that, with time, we will simplify the presentation.

This is in keeping with the rest of set theory; this was always intended as a technical way to implement mathematical ideas.

The underlying mathematical idea of our substitution action is simple. We will conclude with some examples.

Substitution

$a[a{\mapsto}x] = x \qquad b[a{\mapsto}x] = x$

$\{a, b\}[a{\mapsto}x] = \{x, b\}$.

$\mathbb{A}[a{\mapsto}x] = \{a, b, c, d, \ldots\} = \mathbb{A}$ ($a\#\mathbb{A}$).

(So we don't just 'replace a by x in $\mathbb{A}$'.

$\{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \ldots\}[a{\mapsto}\{a, b, c\}] =$
$\{\{\{a, b, c\}, d\}, \{\{a, b, c\}, e\}, \ldots\}$.

Note the capture-avoidance here; we drop $\{a, b\}$ and $\{a, c\}$ because the b and c clash with the b and c in the support of $\{a, b, c\}$.

Substitution

$\{\{a\},\{b\},\{c\},\{d\},\{e\},\ldots\}[a\mapsto\{a,b,c\}] =$
$\{\{a\},\{b\},\{c\},\{d\},\{e\},\ldots\}$.

($a\#\{\{a\},\{b\},\{c\},\{d\},\{e\},\ldots\}$)

Substitution

$(\mathbb{A} \cup \{\{b\}\})[a \mapsto \{a, b, c\}] = \mathbb{A} \cup \{\{b\}\}$.

($a \# \mathbb{A} \cup \{\{b\}\}$)

$(\mathbb{A} \cup \{\{b\}\})[b \mapsto \{a, b, c\}] = \mathbb{A} \cup \{\{\{a, b, c\}\}\}$.

We do not touch the $b \in \mathbb{A}$, because $b \# \mathbb{A}$.

Future work

It is natural to give semantics to propositional logic using sets; the denotation of a proposition is a set.

In FM sets, sets have a support of atoms, which we interpret as being the arbitrary elements they depend on. Substitution only affects atoms in the support.

So we should be able to give semantics to predicate logic using FM sets; the denotation of a predicate is a set; the free variables of the predicate are reflected in how atoms occur in that set.

$\forall a.x$ is just $\bigcap\{x[a\mapsto y]\}$ for some suitable collection of y.

$\exists a.x$ is just $\bigcup\{x[a\mapsto y]\}$ for some suitable collection of y.

We also believe a similar treatment is possible for the λ-calculus.